WORKING PAPER

On the concavity of the consumption function with liquidity constraints

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On the Concavity of the Consumption Function with Liquidity Constraints[*]

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Abstract

Carroll and Kimball (1996) prove that the consumption function is concave if infinitely-lived risk-averse households have a utility function which exhibits Hyperbolic Absolute Risk Aversion (HARA), face income uncertainty, and are prudent. However, the empirical evidence is inconclusive about the importance of income uncertainty for households. On the other hand, empirical results suggest that liquidity and liquidity constraints are important determinants of household behavior. In this paper, I prove that the consumption function is strictly concave in wealth for infinitely-lived risk-averse households with a utility function of the HARA class if there exists a liquidity constraint which binds for some level of wealth. This result is independent of prudence. Furthermore, the introduction of a liquidity constraint always reduces consumption and increases the marginal propensity to consume out of wealth.

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1 Introduction

Carroll and Kimball (1996) prove that the consumption function is concave in wealth if households have a utility function which exhibits Hyperbolic Absolute Risk Aversion (HARA), face income uncertainty, and are prudent.[1] However, there is much disagreement about the empirical relevance of income uncertainty for precautionary saving. Although models of income fluctuations predict that households facing higher income risks should save more, the empirical literature is inconclusive. Kazarosian (1997) and Carroll and Samwick (1997, 1998) find evidence of large precautionary savings, while Guiso, Jappelli, and Terlizzese (1992), Lusardi (1998), and Fulford (2015) find almost no such effects.[2] Moreover, Jappelli, Padula, and Pistaferri (2008) test directly whether households use savings self-insure against income fluctuations and reject the model.

The importance of liquidity and liquidity constraints for household behavior, on the other hand, is well established. Leth-Petersen (2010) use an exogenous change in access to credit in Denmark and Aydın (2015) use randomized trial on credit card limits and both show that households spend more if their credit limits are extended. Furtermore, Baker (2015) show that households' consumption is more sensitive to income changes when their debt-to-asset ratio is high and Fagereng, Holm, and Natvik (2016) show that the marginal propensity to consume of Norwegian lottery winners is decreasing in the amount of liquid reserves. Liquidity constraints therefore seem to be at the heart of models of household behavior.

This paper provides, to my knowledge, the first rigorous analytical proof of the effects of a liquidity constraint on optimal consumption for a general class of utility functions. I prove that the consumption function is strictly concave in wealth for infinitely-lived

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[1]Carroll and Kimball (1996) define prudence as $\frac{u'''u'}{(u'')^2} > 0$. For a risk-averse household with positive marginal utility, this is satisfied if $u''' > 0$.

[2]The empirical literature struggles with finding exogenous changes in household risk. This may present a positive bias on the results since riskier occupations may attract individuals who are less concerned about risk. For example, Hurst, Lusardi, Kennickell, and Torralba (2010) find that business owners who often face higher income volatility have higher observed wealth.

risk-averse households with a utility function of the HARA class if there exists a liquidity constraint and the rate of time-discounting is greater than the rate of interest. Moreover, the presence of a liquidity constraint results in lower consumption and higher marginal propensities to consume (MPC) out of wealth for all households. The main implication is that a liquidity constraint has the same qualitative effects on the consumption function as prudence and income risk (see Kimball, 1990a,b; Carroll and Kimball, 1996), but the effects of a liquidity constraint are independent of both prudence[3] and uncertainty.

The key to my proof relies on the observation that the household problem is a solvable first-order differential equation in the marginal value function in the presence of a liquidity constraint. Using this framework, I derive an implicit function for optimal consumption.[4] The liqudity constraint acts as a boundary constraint for consumption, imposing the condition that consumption must be less than or equal to total income at the liquidity constraint. If the real interest rate is lower than the rate of time discounting, the liquidity constraint binds at the constraint and introduces a shadow cost on their optimal consumption function. Households respond to this constraint by reducing consumption at the liquidity constraint, but they also reduce consumption for all levels of wealth. This reduction in consumption introduces both higher marginal propensities to consume out of wealth and a strictly concave consumption function.

The effects of liquidity constraints on consumption has been analyzed analytically in more specialized settings. Carroll and Kimball (2001) show that for either quadratic, CRRA or CARA utility, a liquidity constraint makes the consumption function concave in wealth in the neighbourhood of the liquidity constraint. Fernández-Corugedo (2002) proves that the consumption function is concave in the presence of soft liquidity constraints, in the sense of penalty costs, although the conditions are stricter than mine.[5] Park (2006)

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[3]As emphasized by Huggett and Ospina (2001) and Huggett (2004), and shown in the special case of quadratic utility by Nishiyama and Kato (2012).

[4]Cao and Werning (2015) solves a similar problem for CRRA utility and find the same implicit function for consumption, see their online appendix pp. 14-15.

[5]Fernández-Corugedo (2002) proves the results for HARA utility, but imposes the constraint that $u''' \geq 0$.

emphasizes the fact that the consumption function is affected by the liquidity constraint, even for levels of wealth far from the liquidity constraint, suggesting that the liquidity constraint might have far-reaching consequences for consumption. Moreover, Nishiyama and Kato (2012) show that the consumption function is concave in wealth if a liqudiity constraint exists and the utility function is quadratic, pointing out that prudence is not a necessary condition for the consumption function to be concave. The main contribution of this paper is therefore to provide a rigorous proof of how hard liquidity constraints affect consumption for a general class of utility functions and in the absence of prudence and uncertainty.

2 The model

The model is a Huggett (1993) economy with wealth taking the form of unproductive bonds. There is a continuum of ex ante identical infinitely-lived households who are heterogeneous in their wealth holdings a_t and have the same income y. Households maximize their lifetime utility flow from future consumption c_t discounted at rate $\rho \geq 0$

$$E_t \int_t^\infty e^{-\rho s} u(c(a_s)) ds \tag{1}$$

where $u(c)$ is a utility function of the HARA class.[6] Household wealth takes the form of risk-free bonds and evolve according to

$$da_t = (ra_t + y - c(a_t))dt \tag{2}$$

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[6]The general utility function from the HARA class can be expressed as $u(c) = \frac{1-\beta_2}{\beta_2} \left(\frac{\beta_1}{1-\beta_2} c + \beta_3 \right)^{\beta_2}$ with $\beta_1 > 0$ and $\beta_3 > -\frac{\beta_1}{1-\beta_2} c$.

where r is the real interest rate. Households face an exogenous liquidity constraint on wealth holdings

$$a_t \geq \underline{a} \tag{3}$$

where $\underline{a}$ is a scalar which satisfies $\underline{a} > -\frac{y}{r}$.

For the rest of this paper, I analyze the household problem in a stationary setting and present the equations without time subscripts. The stationary equilibrium can be summarized by the following Hamilton-Jacobi-Bellman (HJB) equation[7]

$$\rho V(a) = \max_c u(c) + V_a(a)(ra + y - c(a)) \tag{4}$$

where $V(a)$ is the value function of a household with assets a and V_i is the partial derivative operator. The first order necessary condition of the HJB-equation is

$$u'(c) = V_a(a) \tag{5}$$

By applying the envelope condition on the HJB-equation and using the first order condition to replace for consumption, the optimal solution to the household problem can be expressed as a first-order non-linear differential equation in the marginal value function

$$(u')^{-1}(V_a(a)) = ra + y - \frac{V_a(a)}{V_{aa}(a)}(\rho - r) \tag{6}$$

where $(u')^{-1}$ is the inverse of marginal utility.[8]

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[7]See Achdou, Han, Lasry, Lions, and Moll (2016) for a derivation.

[8]Note that equation (6) hinges on the assumption that the utility function is invertible. The HARA utility function is always invertible since $\beta_1 > 0$.

3 Liquidity constraints and consumption

Theorem 1 presents the main result on the effects of a liquidity constraint on the consumption function. For utility functions in the HARA class with $u' > 0$ and $u'' \leq 0$, if a liquidity constraint exists and $\rho > r > 0$, then the liquidity constraint has three effects on the optimal consumption function. First, the presence of a liquidity constraint reduces consumption for all levels of wealth. The liquidity constraint imposes the condition that consumption has to be no greater than income at the liquidity constraint. Since the constraint binds when $\rho > r$, consumption has to be lower at the liquidity constraint than in the unconstrained case. Since households smooth consumption, this costly reduction in consumption at the constraint also reduces consumption for all other levels of wealth.

Second, the marginal propensity to consume out of wealth is always greater with a liquidity constraint than in the unconstrained case. Since the effect of the liquidity constraint is decreasing in the distance from the liquidity constraint, the reduction in consumption above is greatest for households close to the constraint and decreasing in the level of wealth. The optimal consumption function is therefore more steep in the wealth direction when a liquidity constraint exists, resulting in an increase in the marginal propensity to consume out of wealth.

Third, the optimal consumption function is strictly concave in wealth in the presence of a liquidity constraint, even in the absence of income risk. The effects of the liquidity constraint are decreasing in the distance from the liquidity constraint. The reduction in consumption as a result of the constraint is therefore most for households close to the constraint and the effects gradually decline as households move away from the constraint. The resulting optimal consumption function is strictly concave in wealth. Note that the result does not rely on prudence, but holds for a general class of utility functions (HARA) as long as $u' > 0$ and $u'' < 0$, conditions that are satisfied for most utility specifications used in the literature, including CRRA, CARA, and quadratic utility.

Theorem 1. *For utility functions of the HARA class with $u' > 0$ and $u'' < 0$, if a liquidity*

constraint exists with $\underline{a} > -\frac{y}{r}$, $\rho > r > 0$, and $\frac{\rho - r\beta_2}{1-\beta_2} > 0$. Then there exists a unique optimal consumption function for all $a \in (\underline{a}, \infty)$ with

(i) *Consumption is lower than in the unconstrained case, $c(a) < c^u(a)$.*

(ii) *The MPC out of wealth is greater than in the unconstrained case, $c_a(a) > c_a^u(a)$.*

(iii) *Consumption is strictly concave in wealth, $c_{aa}(a) < 0$.*

Theorem 1 implies that a liquidity constraint and the presence of temporary income risk have the same qualitative effects on the optimal consumption function. Kimball (1990a) and Carroll and Kimball (1996) show that for utility functions of the HARA class with $u' > 0$, $u'' < 0$, and $u''' \geq 0$, then transitory income risk reduces consumption, increases the marginal propensity to consume out of wealth and introduces concavity in the wealth direction to the optimal consumption function. A liquidity constraint has the same qualitative effects on the optimal consumption function. We can therefore conclude that neither liquidity constraints nor income risk is necessary for the optimal consumption function to be concave in wealth, but both are sufficient.

Remark 1. *The results in Theorem 1 only holds for $a \in (\underline{a}, \infty)$. If $a = \underline{a}$, then $c(\underline{a}) = r\underline{a} + y$ while $c_a(\underline{a})$ and $c_{aa}(\underline{a})$ are undefined.* [9]

Remark 2. *The condition $\underline{a} > -\frac{y}{r}$ ensures that there is no bankruptcy and that the household can always pay the interest rate cost on assets (natural borrowing limit).*

Remark 3. *Theorem 1 is independent of prudence, $u'''(c)$.*

Remark 4. *The condition $\rho > r > 0$ ensures that the liquidity constraint is relevant and affects consumption.*

Remark 5. *The condition $\frac{\rho - r\beta_2}{1-\beta_2} > 0$ ensures that consumption is an increasing function of assets in the unconstrained case. The condition holds as long as $\beta_2 \notin [1, \frac{\rho}{r})$.*

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[9]One can show that $\lim_{a \to \underline{a}} c_a(a) = \infty$ and $\lim_{a \to \underline{a}} c_{aa}(a) = -\infty$. A proof of this is available upon request. However, Lemma 1 does not hold for $a = \underline{a}$ so uniqueness is not ensured.

Proof. The proof of Theorem 1 contains four steps. First, I show in Lemma 1 that there exists a unique marginal value function which solves equation (6). Second, I provide the analytical solution to equation (6) with a liquidity constraint in Lemma 2. The key observation is that the optimal solution contains one free parameter. This parameter is pinned down by the liquidity constraint which acts as a boundary constraint for the household problem. As a result, the liquidity constraint explicitly enters the household problem and I can derive the effects of this constraint. Third, in order to describe the effects of the constraint, I have to compare the solution with the optimal solution without a constraint. In Lemma 3, I derive the linear consumption function in the unconstrained case. Fourth, the results in Theorem 1 then follows by applying repeated total derivation on the constrained and the unconstrained consumption functions and comparing the these.

The first step is to show existence and uniqueness of an optimal solution to the household problem. Lemma 1 shows that since the household problem is bounded below in wealth by the liquidity constraint, there exists a unique marginal value function which solves equation (6).

Lemma 1. *Existence and uniqueness.*

If $u' > 0$, $u'' < 0$, a liquidity constraint exists with $\underline{a} > -\frac{y}{r}$, and $\rho > r > 0$, then there exists a unique $V_a(a)$ which solves equation (6) for all $a \in (\underline{a}, \infty)$.

Proof. See appendix A.1. □

The second step is to solve for the closed form solution of the optimal value function. I show in Lemma 2 that equation (6) is in fact solvable and that the solution contains a free parameter which is pinned down by the liquidity constraint.

Lemma 2. *The analytical solution with no income risk and HARA utility.*

For utility functions of the HARA class with

$$u'(c) = \beta_1 \left(\frac{\beta_1 C}{1 - \beta_2} + \beta_3 \right)^{\beta_2 - 1}$$

then equation (6) *has the unique analytical solution*

$$ra + y = \frac{(1 - \beta_2)^2 r}{\beta_1(\rho - r\beta_2)} \left(\frac{V_a(a)}{\beta_1} \right)^{\frac{1}{\beta_2 - 1}} - \frac{\beta_3(1 - \beta_2)}{\beta_1} + A_0 V_a(a)^{\frac{r}{\rho - r}} \tag{7}$$

where A_0 is a constant defined by the liquidity constraint

$$u'(r\underline{a} + y) \geq V_a(\underline{a}) \tag{8}$$

Proof. See appendix A.2. □

The solution to the household problem consists of two equations, the analytical expression for the value function (7) and the liquidity constraint (8). I use the liquidity constraint (8) to pin down the constant A_0. Moreover, given that the first order condition is a mapping between the marginal value function and consumption, the solution in Lemma 2 also yields a unique solution for the optimal consumption function.

The third step relies on deriving the optimal solution to the unconstrained household problem. Given Lemma 2, there exists a special case where $A_0 = 0$ which yields a linear consumption function. Since $A_0 = 0$ is equivalent to stating that the liquidity constraint never binds, this is the unconstrained solution to the household problem.

Lemma 3. *The unconstrained consumption function, $c_u(a)$.*

For utility functions of the HARA class with

$$u'(c) = \beta_1 \left(\frac{\beta_1 C}{1 - \beta_2} + \beta_3 \right)^{\beta_2 - 1}$$

then there exists a consumption function which is linear in assets

$$c^u(a) = \xi + \xi_a a$$

where the coefficients are given by

$$\xi = \frac{\rho - r\beta_2}{(1 - \beta_2)r}y + \frac{\beta_3(\rho - r)}{\beta_1 r} \tag{9}$$

$$\xi_a = \frac{\rho - r\beta_2}{1 - \beta_2} \tag{10}$$

Proof. See appendix A.3. □

The fourth step of the proof follows by applying repeated total derivation on the constrained solution for consumption from Lemma 2 and the unconstrained solution in Lemma 3 and comparing these (see appendix A.4). □

3.1 An illustration of Theorem 1

In order to provide some intuition behind the results in Theorem 1, I illustrate how the liquidity constraint affects consumption for a power utility function (CRRA) in this section.

Example 1. *The analytical solution for CRRA utility.*
For CRRA utility with $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, *we have* $\beta_1 = 1 - \beta_2$, $\beta_2 = 1 - \gamma$, $\gamma \geq 0$, *and* $\beta_3 = 0$. *Then the analytical solution to the consumption function is*

$$ra + y = \frac{\gamma r}{\gamma r - r + \rho}c(a) + A_0 c(a)^{\frac{\gamma r}{r - \rho}}$$

where A_0 *is defined by the liquidity constraint*

$$A_0 = \left(\frac{\rho - r}{\gamma r - r + \rho}\right)(r\underline{a} + y)^{\frac{\gamma r}{\rho - r} + 1}$$

We can compare the consumption function with and without a liquidity constraint by comparing the solution with $A_0 > 0$ to the solution with $A_0 = 0$ (linear unconstrained case, Lemma 3). Figure 1 illustrates how the liquidity constraint affects consumption when income is normalized to 1. The blue lines represent consumption and saving in the unconstrained case with $\rho = r$ where consumption is equal to income and savings is zero. The red lines illustrate how changes in r affect consumption and savings in the unconstrained case. A reduction in r shifts consumption down and savings up. Furthermore, it affects the slopes since r is the return on capital and directly affects income.
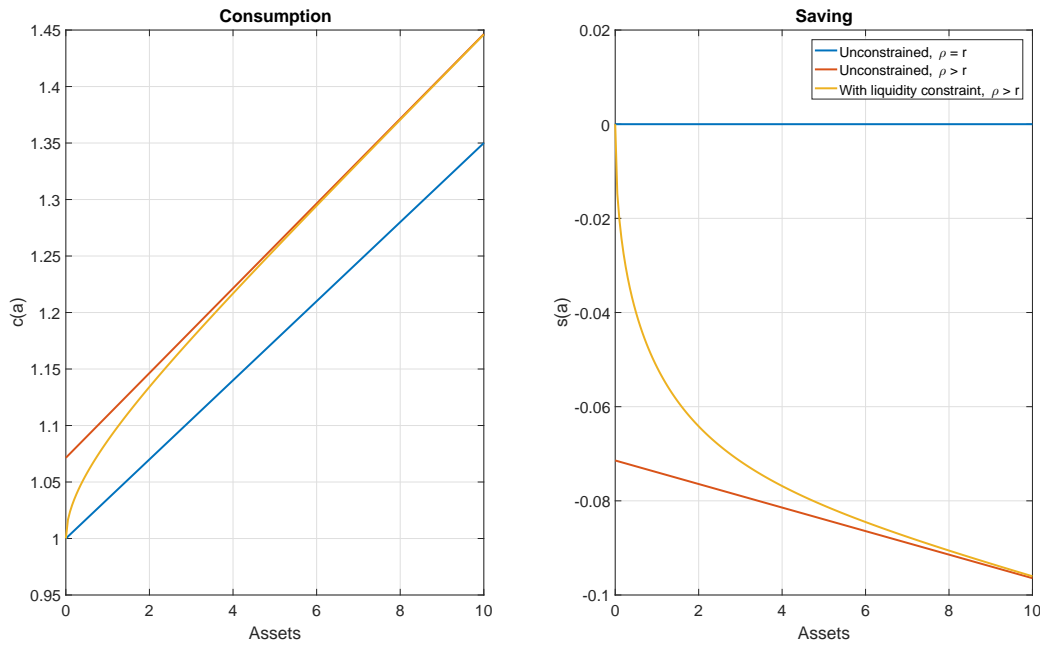
Figure 1: The effects of liquidity constraints in the CRRA case. Calibration: $\gamma = 2$; $y = 1$, $\rho = 0.04$, $r = 0.04$ (blue) or 0.035 (red) or 0.035 (orange), and $\underline{a} = 0$ (orange).

The orange lines represent consumption and saving with a liquidity constraint and with the same calibration as the red case. The difference between the red and orange lines illustrates how the liquidity constraint affects consumption and savings. First, the liquidity constraint imposes the condition that savings has to be non-negative at the

liquidity constraint. Since saving is negative in the unconstrained case (red line), saving shifts up and consumption shifts down. Furthermore, saving is not only affected at the liquidity constraint, but also for all other levels of wealth. The effects of the liquidity constraint are decreasing in wealth, as observed by the decreasing distance between the red and orange lines as wealth increases. This diminishing effect of the liquidity constraint implies that consumption is concave in wealth. All three results in Theorem 1 are visible in Figure 1: consumption decreases, the marginal propensity to consume out of wealth increases, and the optimal consumption function is concave in wealth.

3.2   Concavity is independent of prudence

In order to illustrate how the result holds for all types of HARA utility functions, also for utility functions with negative prudence, $u''' < 0$, I provide an example of a consumption function for each major type of utility function in the HARA class. Figure 2 shows four examples of consumption functions for the four types of HARA utility functions: CARA, CRRA, quadratic utility, and negative prudence. All consumption functions are concave when there exists a liquidity constraint and this result holds even for utility functions with no prudence (quadratic) and negative prudence.[10]

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[10]In the example, I use the utility function $u(c) = -\frac{1}{3}(-2c + 10)^{\frac{3}{2}}$ where $u'''(c) = -(-2c + 10)^{-\frac{1}{2}} < 0$.
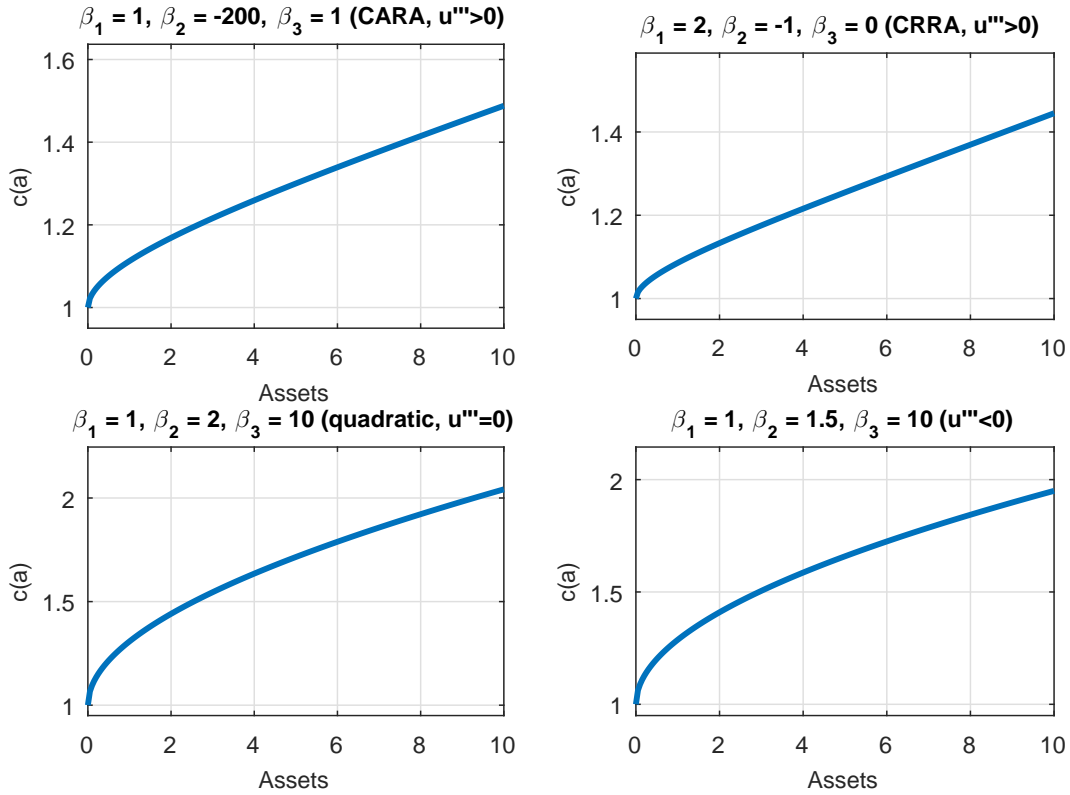
Figure 2: The consumption function for HARA class utility functions. Calibration: $y = 1$, $\rho = 0.04$, $r = 0.035$, and $\underline{a} = 0$.

4 A generalization

In this section, I generalize the results in the previous section to the case where the liquidity constraint changes. The main observation is that the degree to which consumption is affected by a liquidity constraint is governed by a simple metric, the distance between savings at the liquidity constraint in the unconstrained case and zero (the liquidity constrained case). I define this metric as the cost of the liquidity constraint[11] and analyze the effects of changes in this cost on consumption.

Definition 1. *The cost of the liquidity constraint*

The cost of the liquidity constraint is defined as the non-negative distance between saving at the

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[11]The cost of the liquidity constraint has the same interpretation as the shadow cost of the constraint.

liquidity constraint in the unconstrained case, $s^u(\underline{a})$ and zero (the constrained case)

$$\kappa = \max\{0, s(\underline{a}) - s^u(\underline{a})\} = \max\{0, -s^u(\underline{a})\}$$

The advantage of defining the concept of the cost of the liquidity constraint is that it represents many possible policies that might affect the liquidity constraint. For example, an increase in the cost of the liquidity constraint could be induced by tighter of credit conditions, less income risk in the sense of lower levels of precautionary savings,[12] or a reduction in the real interest rate. The method remains agnostic to the cause of the change in the cost of the liquidity constraint, but allows me to analyze the effects.

Theorem 2 presents three qualitative effects of an increase in the cost of the liquidity constraint on the optimal consumption function: consumption decreases for all levels of wealth,[13] the marginal propensity to consume out of wealth increases, and the consumption function becomes more concave in wealth in the sense that the second derivative of the consumption function with respect to wealth decreases. The implication of Theorem 2 is that the results in Theorem 1 also holds for changes in the cost of the liquidity constraint. A higher cost the liquidity constraint always strengthen the effects of the liquidity constraint, inducing a greater deviation from the unconstrained case.

Theorem 2. *The effects of changes in the cost of the liquidity constraint on consumption*

For utility functions of the HARA class, if $u' > 0$, $u'' < 0$, a liquidity constraint exists with $\underline{a} > -\frac{y}{r}$, $\rho > r > 0$, and $\frac{\rho - r\beta_2}{1 - \beta_2} > 0$. Then for all $a \in (\underline{a}, \infty)$, the effects of an increase in the cost of the liquidity constraint $\hat{\kappa} > \kappa$ on consumption are

(i) *Consumption decreases, $c(a; \hat{\kappa}) < c(a; \kappa)$.*

(ii) *The MPC out of wealth increases, $c_a(a; \hat{\kappa}) > c_a(a; \kappa)$.*

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[12]Less income risk reduces saving for all levels of wealth, also at the liquidity constraint, thus increasing the cost of the liquidity constraint.

[13]Theorem 2.i is closely related to Theorem 9 in Chamberlain and Wilson (2000). Chamberlain and Wilson (2000) analyze a consumption function with a sequence of liquidity constraints and income risk. They find that a less restrictive sequence of liquidity constraints implies that wealth accumulation is lower, i.e. consumption increases if the liquidity constraints are less restrictive.

If also $\left(\frac{\rho - r\beta_2}{1 - \beta_2} + \frac{\rho - r}{1 - \beta_2} \right) > 0$, then

   (iii) The consumption function is more concave in wealth, $c_{aa}(a; \hat{\kappa}) < c_{aa}(a; \kappa)$.

    The concept of the cost of the liquidity constraint is closely related the discussion on debt constraints vs. liquidity constraints. Kehoe and Levine (2001) compare two models of market incompleteness, one model where consumers have a single asset that they cannot sell short (liquidity constraint) and one model where consumers cannot borrow so much that they would want to default (debt constraint). They show that the debt constraint model has less effects on the dynamics of the model (compared to complete markets) since it is a smaller departure from the complete markets framework. In my framework, comparing a debt constraint and a liquidity constraint is the same as comparing costs of liquidity constraints. The debt constraint inflicts a lower cost of the liquidity constraint. Subsequently, it has less effect on the curvature of the consumption function and less implications for dynamics. The cost of the liquidity constraint can therefore be interpreted as a measure of the deviation from complete markets.

Remark 6. *The condition $\left(\frac{\rho - r\beta_2}{1 - \beta_2} + \frac{\rho - r}{1 - \beta_2} \right) > 0$ in Theorem 2 holds if $\beta_2 \notin [1, \frac{2\rho}{r} - 1)$. Note that this condition still allows strictly negative prudence as long as $\beta_2 \in [\frac{2\rho}{r} - 1, 2)$*

Proof. The proof of Theorem 2 relies on the observation that the cost of the liquidity constraint, κ is an increasing function of A_0.

Lemma 4. *For utility functions of the HARA class, if $u' > 0$, $u'' < 0$, a liquidity constraint exists with $\underline{a} > -\frac{y}{r}$, $\rho > r > 0$, and $\frac{\rho - r\beta_2}{1 - \beta_2} > 0$. Then A_0 is an increasing function of κ.*

Proof. See appendix A.5   □

The results in Theorem 2 follow by applying total derivation with respect to A_0 on the implicit expressions for consumption, marginal propensity to consume, and concavity; see appendix A.6.   □

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A Proofs

A.1 Proof of Lemma 1

Proof. Rearrange equation (6)

$$V_{aa}(a) = \frac{\rho - r}{ra + y - (u')^{-1}(V_a(a))} V_a(a)$$

then we have

$$0 < V_a(a) = u'(c(a)) \leq u'(r\underline{a} + y) < \infty$$

where the first inequality follows because $u' > 0$ by assumption and the equality follows from the first order condition. The second inequality follows from the fact that $u'' \leq 0$ ($r\underline{a} + y$ is consumption at the liquidity constraint and therefore the highest value of u' since $u'' \leq 0$). The third inequality follows because $ra + y$ is strictly positive since we assumed that $\underline{a} > \frac{y}{r}$.

Furthermore, we have that

$$\left| \frac{\rho - r}{ra + y - c(a)} \right| < \infty$$

where the inequality follows because as long as $\rho > r$, then consumption is strictly greater than income for $a > \underline{a}$. Thus, $c(a) > ra + y$ and the fraction has a finite absolute value.

We therefore have that

$$|V_{aa}(a)| = \left| \frac{\rho - r}{ra + y - c(a)} V_a(a) \right| < \infty$$

By Körner (2004, Theorem 12.16), there exists a unique solution which satisfies equation (6). □

A.2 Proof of Lemma 2

Proof. I prove Lemma 2 by showing that equation (7) always solves equation (6). The first step is to insert for $ra + y$ from solution equation (7) into the original differential equation (6)

$$\frac{(1-\beta_2)^2 r}{\beta_1(\rho - r\beta_2)}\left(\frac{V_a(a)}{\beta_1}\right)^{\frac{1}{\beta_2-1}} - \frac{\beta_3(1-\beta_2)}{\beta_1} + A_0 V_a(a)^{\frac{r}{\rho-r}} - \frac{1-\beta_2}{\beta_1}\left(\left(\frac{V_a(a)}{\beta_1}\right)^{\frac{1}{\beta_2-1}} - \beta_3\right) = \frac{V_a(a)}{V_{aa}(a)}(\rho - r)$$

Note that β_3 cancels out such that we can remove it from the expression

$$\frac{(1-\beta_2)^2 r}{\beta_1(\rho - r\beta_2)}\left(\frac{V_a(a)}{\beta_1}\right)^{\frac{1}{\beta_2-1}} + A_0 V_a(a)^{\frac{r}{\rho-r}} - \frac{1-\beta_2}{\beta_1}\left(\frac{V_a(a)}{\beta_1}\right)^{\frac{1}{\beta_2-1}} = \frac{V_a(a)}{V_{aa}(a)}(\rho - r) \qquad (11)$$

The second step is to apply total derivation to solution equation (7)

$$r = -\frac{(1-\beta_2)r}{\beta_1(\rho - r\beta_2)}\left(\frac{V_a(a)}{\beta_1}\right)^{\frac{1}{\beta_2-1}}\frac{V_{aa}(a)}{V_a(a)} + \frac{A_0 r}{\rho - r}V_a(a)^{\frac{r}{\rho-r}}\frac{V_{aa}(a)}{V_a(a)}$$

$$\frac{V_a(a)}{V_{aa}(a)} = -\frac{(1-\beta_2)}{\beta_1(\rho - r\beta_2)}\left(\frac{V_a(a)}{\beta_1}\right)^{\frac{1}{\beta_2-1}} + \frac{A_0}{\rho - r}V_a(a)^{\frac{r}{\rho-r}} \qquad (12)$$

Next, insert equation (12) into equation (11)

$$\frac{(1-\beta_2)^2 r}{\beta_1(\rho - r\beta_2)}\left(\frac{V_a(a)}{\beta_1}\right)^{\frac{1}{\beta_2-1}} + A_0 V_a(a)^{\frac{r}{\rho-r}} - \frac{1-\beta_2}{\beta_1}\left(\frac{V_a(a)}{\beta_1}\right)^{\frac{1}{\beta_2-1}} = -\frac{(1-\beta_2)(\rho - r)}{\beta_1(\rho - r\beta_2)}\left(\frac{V_a(a)}{\beta_1}\right)^{\frac{1}{\beta_2-1}} + A_0 V_a(a)^{\frac{r}{\rho-r}}$$

The A_0 expressions cancel and we are left with

$$\left(\frac{V_a(a)}{\beta_1}\right)^{\frac{1}{\beta_2-1}}\left(\frac{(1-\beta_2)^2 r}{\beta_1(\rho - r\beta_2)} - \frac{1-\beta_2}{\beta_1} + \frac{(1-\beta_2)(\rho - r)}{\beta_1(\rho - r\beta_2)}\right) = 0$$

$$\frac{(1-\beta_2)r}{(\rho - r\beta_2)} - 1 + \frac{(\rho - r)}{(\rho - r\beta_2)} = 0$$

$$r - \beta_2 r - \rho + \beta_2 r + \rho - r = 0$$

$$0 = 0$$

Which is what I needed to show. As a result, equation (7) solves equation (6). By Lemma 1, we know that the solution is unique if the liquidity constraint is imposed and Lemma 2 is proved.

The condition for the liquidity constraint:

The liquidity constraint implies that saving has to be non-negative at the liquidity cons-triant. This condition fixes A_0.

$$s(\underline{a}) \geq 0$$

$$r\underline{a} + y - c(\underline{a}) \geq 0$$

$$r\underline{a} + y - (u')^{-1}(V_a(\underline{a})) \geq 0$$

$$u'(r\underline{a} + y) \geq V_a(\underline{a})$$

$\square$

A.3   Proof of Lemma 3

Proof. The starting point is equation (6). I first replace all terms with the first order condition (equation (5)) and obtain a differential equation in consumption

$$ra + y - c(a) = -\frac{u'(c(a))}{u''(c(a))c_a(a)}(r - \rho)$$

For the HARA-class utility functions, we know that

$$-\frac{u'(c(a))}{u''(c(a))} = \frac{1}{1 - \beta_2}c(a) + \frac{\beta_3}{\beta_1}$$

Using this condition, the equation becomes

$$ra + y - c(a) = -\left(\frac{1}{1 - \beta_2}c(a) + \frac{\beta_3}{\beta_1}\right)\frac{1}{c_a(a)}(\rho - r)$$

In order to find a linear solution, I assume that $c(a) = \xi_a a + \xi$. Subsequently, $c_a(a) = \xi_a$ and inserting this into the equation yields

$$ra + y - \xi_a a - \xi = -\left(\phi_1 \xi_a a + \frac{1}{1-\beta_2}\xi + \frac{\beta_3}{\beta_1}\right)\frac{1}{\xi_a}(\rho - r)$$

This yields two equations in the two unknowns ξ_a and ξ

$$r - \xi_a = -\frac{1}{1-\beta_2}(\rho - r)$$

$$y - \xi = -\frac{\frac{1}{1-\beta_2}\xi + \frac{\beta_3}{\beta_1}}{\xi_a}(\rho - r)$$

This system of equations is solvable recursively and gives the two equations in Lemma 3. $\qquad\square$

A.4 Proof of Theorem 1

Proof. In this proof, I show that the consumption function is concave for a general HARA utility function if $u' > 0$ and $u'' \le 0$, $\rho > r > 0$, and $\frac{\rho - r\beta_2}{1-\beta_2} > 0$. From Lemma 2, we know that if the utility function is of the HARA form

$$u'(c) = \beta_1\left[\frac{\beta_1}{1-\beta_2}c + \beta_3\right]^{\beta_2 - 1}$$

where the definition of HARA implies that $\beta_1 > 0$ and $\beta_3 > -\frac{\beta_1}{1-\beta_2}c(a)$.

The general solution to equation 6 is

$$ra + y = \frac{(1-\beta_2)^2 r}{\beta_1(\rho - r\beta_2)}\left(\frac{V_a(a)}{\beta_1}\right)^{\frac{1}{\beta_2 - 1}} - \frac{\beta_3(1-\beta_2)}{\beta_1} + A_0 V_a(a)^{\frac{r}{\rho - r}}$$

The first step in this proof is to use the first order condition (equation 5) to insert for $V_a(a)$ in the equation above

$$ra + y = \frac{(1 - \beta_2)^2 r}{\beta_1(\rho - r\beta_2)} \left[\frac{V_a(a)}{\beta_1} \right]^{\frac{1}{\beta_2 - 1}} - \frac{\beta_3(1 - \beta_2)}{\beta_1} + A_0 V_a(a)^{\frac{r}{\rho - r}}$$

$$ra + y = \frac{(1 - \beta_2)^2 r}{\beta_1(\rho - r\beta_2)} \left[\frac{\beta_1}{1 - \beta_2} c(a) + \beta_3 \right] - \frac{\beta_3(1 - \beta_2)}{\beta_1} + A_0 \beta_1^{\frac{r}{\rho - r}} \left[\frac{\beta_1}{1 - \beta_2} c(a) + \beta_3 \right]^{-\frac{(1 - \beta_2)r}{\rho - r}}$$

$$ra + y = \frac{(1 - \beta_2)r}{\rho - r\beta_2} c(a) - \frac{1 - \beta_2}{\beta_1} \frac{\rho - r}{\rho - r\beta_2} \beta_3 + A_0 \beta_1^{\frac{r}{\rho - r}} \left[\frac{\beta_1}{1 - \beta_2} c(a) + \beta_3 \right]^{-\frac{(1 - \beta_2)r}{\rho - r}} \tag{13}$$

Furthermore, from Lemma 2 we know that the solution for A_0 is defined by $u'(r\underline{a} + y) = V_a(\underline{a})$

$$A_0 = \frac{\rho - r}{\rho - r\beta_2} \frac{1 - \beta_2}{\beta_1^{\frac{\rho}{\rho - r}}} \left[\frac{\beta_1}{1 - \beta_2} (r\underline{a} + y) + \beta_3 \right]^{\frac{\rho - r\beta_2}{\rho - r}}$$

Using the solution for A_0, we insert for A_0 into equation 13 above

$$ra + y = \frac{(1 - \beta_2)r}{\rho - r\beta_2} c(a) - \frac{1 - \beta_2}{\beta_1} \frac{\rho - r}{\rho - r\beta_2} \beta_3 + \frac{\rho - r}{\rho - r\beta_2} \frac{1 - \beta_2}{\beta_1^{\frac{\rho}{\rho - r}}} \left[\frac{\beta_1}{1 - \beta_2} (r\underline{a} + y) + \beta_3 \right]^{\frac{\rho - r\beta_2}{\rho - r}} \beta_1^{\frac{r}{\rho - r}} \left[\frac{\beta_1}{1 - \beta_2} c(a) + \beta_3 \right]^{-\frac{(1 - \beta_2)r}{\rho - r}}$$

$$ra + y = \frac{(1 - \beta_2)r}{\rho - r\beta_2} c(a) - \frac{1 - \beta_2}{\beta_1} \frac{\rho - r}{\rho - r\beta_2} \beta_3 + \frac{\rho - r}{\rho - r\beta_2} \frac{1 - \beta_2}{\beta_1} \left[\frac{\beta_1}{1 - \beta_2} (r\underline{a} + y) + \beta_3 \right]^{\frac{\rho - r\beta_2}{\rho - r}} \left[\frac{\beta_1}{1 - \beta_2} c(a) + \beta_3 \right]^{-\frac{(1 - \beta_2)r}{\rho - r}} \tag{14}$$

which implies that we have the following implicit function for consumption

$$c(a) = \frac{\rho - r\beta_2}{(1 - \beta_2)r} (ra + y) + \frac{\rho - r}{\beta_1 r} \beta_3 - \frac{\rho - r}{\beta_1 r} \left[\frac{\beta_1}{1 - \beta_2} (r\underline{a} + y) + \beta_3 \right]^{\frac{\rho - r\beta_2}{\rho - r}} \left[\frac{\beta_1}{1 - \beta_2} c(a) + \beta_3 \right]^{-\frac{(1 - \beta_2)r}{\rho - r}}$$

The results in Theorem 1 now follows.

Part I: $c(a) < c^u(a)$

Since the unconstrained linear solution is $c^u(a) = \frac{\rho - r\beta_2}{(1 - \beta_2)r} (ra + y) + \frac{\rho - r}{\beta_1 r} \beta_3$, we have the first

result

$$c(a) = c^u(a) - \frac{\rho - r}{\beta_1 r}\left[\frac{\beta_1}{1 - \beta_2}(r\underline{a} + y) + \beta_3\right]^{\frac{\rho - r\beta_2}{\rho - r}}\left[\frac{\beta_1}{1 - \beta_2}c(a) + \beta_3\right]^{-\frac{(1-\beta_2)r}{\rho - r}} < c^u(a)$$

since $\beta_1 > 0$, and $\beta_3 > -\frac{\beta_1}{1-\beta_2}c(a)$, and $\rho > r > 0$ by assumption.

Part II: $c_a(a) > c_a^u(a)$

The next step is to do total derivation on the consumption equation.

$$c_a(a) = \frac{\rho - r\beta_2}{1 - \beta_2} + \left[\frac{\beta_1}{1 - \beta_2}(r\underline{a} + y) + \beta_3\right]^{\frac{\rho - r\beta_2}{\rho - r}}\left[\frac{\beta_1}{1 - \beta_2}c(a) + \beta_3\right]^{-\frac{\rho - r\beta_2}{\rho - r}} c_a(a)$$

$$c_a(a)\left[1 - \left[\frac{\frac{\beta_1}{1-\beta_2}(r\underline{a} + y) + \beta_3}{\frac{\beta_1}{1-\beta_2}c(a) + \beta_3}\right]^{\frac{\rho - r\beta_2}{\rho - r}}\right] = \frac{\rho - r\beta_2}{1 - \beta_2}$$

Now, defining β_0 as

$$\beta_0 = \left[\frac{\frac{\beta_1}{1-\beta_2}(r\underline{a} + y) + \beta_3}{\frac{\beta_1}{1-\beta_2}c(a) + \beta_3}\right]^{\frac{\rho - r\beta_2}{\rho - r}}$$

then we know that

$$0 < \beta_0 = \left[\frac{\frac{\beta_1}{1-\beta_2}(r\underline{a} + y) + \beta_3}{\frac{\beta_1}{1-\beta_2}c(a) + \beta_3}\right]^{\frac{\rho - r\beta_2}{\rho - r}} < \left[\frac{\frac{\beta_1}{1-\beta_2}(r\underline{a} + y) + \beta_3}{\frac{\beta_1}{1-\beta_2}(r\underline{a} + y) + \beta_3}\right]^{\frac{\rho - r\beta_2}{\rho - r}} = 1$$

since $\beta_3 > -\frac{\beta_1}{1-\beta_2}c(a)$ and the assumptions that $\frac{\rho - r\beta_2}{1-\beta_2} > 0$ and $\rho > r > 0$ imply that the identity holds ($\frac{\rho - r\beta_2}{1-\beta_2} > 0$ implies that if $\beta_2 > 1$, then $\rho - r\beta_2 < 0$ such that the fraction is inverted). As a result

$$c_a(a) = \frac{1}{1 - \beta_0}\frac{\rho - r\beta_2}{1 - \beta_2} > \frac{\rho - r\beta_2}{1 - \beta_2} = c_a^u(a)$$

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since $\frac{\rho - r\beta_2}{1 - \beta_2} > 0$ by assumption and $0 < \beta_0 < 1$.

Part III: $c_{aa}(a) < 0$

In order to prove the results for concavity, we do another total derivation.

$$c_{aa}(a) = -\frac{\rho - r\beta_2}{\rho - r}\frac{\beta_1}{1 - \beta_2}\left[\frac{\beta_1}{1 - \beta_2}(r\underline{a} + y) + \beta_3\right]^{\frac{\rho - r\beta_2}{\rho - r}}\left[\frac{\beta_1}{1 - \beta_2}c(a) + \beta_3\right]^{-\frac{\rho - r\beta_2}{\rho - r} - 1}(c_a(a))^2$$

$$+ \left[\frac{\beta_1}{1 - \beta_2}(r\underline{a} + y) + \beta_3\right]^{\frac{\rho - r\beta_2}{\rho - r}}\left[\frac{\beta_1}{1 - \beta_2}c(a) + \beta_3\right]^{-\frac{\rho - r\beta_2}{\rho - r}}c_{aa}(a)$$

Moving the second expression to the left-hand side

$$c_{aa}(a)\left[1 - \beta_0\right] = -\frac{\rho - r\beta_2}{1 - \beta_2}\frac{\beta_1}{\rho - r}\left[\frac{\beta_1}{1 - \beta_2}(r\underline{a} + y) + \beta_3\right]^{\frac{\rho - r\beta_2}{\rho - r}}\left[\frac{\beta_1}{1 - \beta_2}c(a) + \beta_3\right]^{-\frac{\rho - r\beta_2}{\rho - r} - 1}(c_a(a))^2$$

$$c_{aa}(a) = -\frac{1}{1 - \beta_0}\frac{\rho - r\beta_2}{1 - \beta_2}\frac{\beta_1}{\rho - r}\left[\frac{\beta_1}{1 - \beta_2}(r\underline{a} + y) + \beta_3\right]^{\frac{\rho - r\beta_2}{\rho - r}}\left[\frac{\beta_1}{1 - \beta_2}c(a) + \beta_3\right]^{-\frac{\rho - r\beta_2}{\rho - r} - 1}(c_a(a))^2$$

$$c_{aa}(a) = -\frac{1}{1 - \beta_0}\frac{\rho - r\beta_2}{1 - \beta_2}\frac{\beta_1}{\rho - r}\left[\frac{\frac{\beta_1}{1 - \beta_2}(r\underline{a} + y) + \beta_3}{\frac{\beta_1}{1 - \beta_2}c(a) + \beta_3}\right]^{\frac{\rho - r\beta_2}{\rho - r}}\frac{1}{\frac{\beta_1}{1 - \beta_2}c(a) + \beta_3}(c_a(a))^2$$

$$c_{aa}(a) = -\frac{\beta_0}{1 - \beta_0}\frac{\rho - r\beta_2}{1 - \beta_2}\frac{\beta_1}{\rho - r}\frac{1}{\frac{\beta_1}{1 - \beta_2}c(a) + \beta_3}(c_a(a))^2 < 0$$

where the last inequality follows since $0 < \beta_0 < 1$, $\frac{\rho - r\beta_2}{1 - \beta_2} > 0$, $\beta_3 > -\frac{\beta_1}{1 - \beta_2}c(a)$, and $\rho > r > 0$ by assumption. $\square$

A.5 Proof of Lemma 4

Proof. In order to prove that A_0 is an increasing function of κ, I start with the general solution under HARA

$$ra + y = \frac{(1 - \beta_2)^2 r}{\beta_1(\rho - r\beta_2)}\left(\frac{V_a(a)}{\beta_1}\right)^{\frac{1}{\beta_2 - 1}} - \frac{\beta_3(1 - \beta_2)}{\beta_1} + A_0 V_a(a)^{\frac{r}{\rho - r}}$$

and insert from the first order condition such that it is defined in terms of consumption

$$ra + y = \frac{(1 - \beta_2)r}{\rho - r\beta_2}c(a) - \frac{1 - \beta_2}{\beta_1}\frac{\rho - r}{\rho - r\beta_2}\beta_3 + A_0 V_a(a)^{\frac{r}{\rho-r}}$$

Now, define consumption in the unconstrained case

$$c^u(a) = \frac{\rho - r\beta_2}{(1 - \beta_2)r}\left[r\underline{a} + y + \frac{1 - \beta_2}{\beta_1}\frac{\rho - r}{\rho - r\beta_2}\beta_3\right]$$

which leads to the following definition of κ

$$\kappa = -s^u(\underline{a}) = c^u(\underline{a}) - r\underline{a} - y = \frac{\rho - r\beta_2}{(1 - \beta_2)r}\left[r\underline{a} + y + \frac{1 - \beta_2}{\beta_1}\frac{\rho - r}{\rho - r\beta_2}\beta_3\right] - r\underline{a} - y$$

Rearrange the equation in terms of $c(a)$ and evaluate the equation at the liquidity constraint

$$c(\underline{a}) = \frac{\rho - r\beta_2}{(1 - \beta_2)r}\left[r\underline{a} + y + \frac{1 - \beta_2}{\beta_1}\frac{\rho - r}{\rho - r\beta_2}\beta_3\right] - A_0\frac{\rho - r\beta_2}{(1 - \beta_2)r}V_a(\underline{a})^{\frac{r}{\rho-r}}$$

Impose the liquidity constraint to solve for A_0

$$r\underline{a} + y = \frac{\rho - r\beta_2}{(1 - \beta_2)r}\left[r\underline{a} + y + \frac{1 - \beta_2}{\beta_1}\frac{\rho - r}{\rho - r\beta_2}\beta_3\right] - A_0\frac{\rho - r\beta_2}{(1 - \beta_2)r}V_a(\underline{a})^{\frac{r}{\rho-r}}$$

then

$$A_0 = \kappa\frac{(1 - \beta_2)r}{\rho - r\beta_2}V_a(\underline{a})^{-\frac{r}{\rho-r}} = \kappa\frac{(1 - \beta_2)r}{\rho - r\beta_2}u'(r\underline{a} + y)^{-\frac{r}{\rho-r}}$$

and A_0 is an increasing function of κ since $u' > 0$, $\rho > r > 0$, and $\frac{\rho - r\beta_2}{1 - \beta_2} > 0$ by assumption.

$\square$

A.6   Proof of Theorem 2

Proof. Since κ is an increasing function of A_0 (from Lemma 4), I show all results for A_0, keeping in mind that the results also hold for κ. From the calculation in the proof of

Theorem 1, we know that consumption is

$$c(a) = \frac{\rho - r\beta_2}{(1-\beta_2)r}(ra + y) + \frac{\beta_1 r}{\rho - r}\beta_3 - A_0\frac{\rho - r\beta_2}{(1-\beta_2)r}\beta_1^{\frac{r}{\rho-r}}\left[\frac{\beta_1}{1-\beta_2}c(a) + \beta_3\right]^{-\frac{(1-\beta_2)r}{\rho-r}}$$

Part I: $\frac{dc(a)}{dA_0} < 0$

Applying total derivation of the consumption expression with respect to A_0 gives

$$\frac{dc(a)}{dA_0} = -\frac{\rho - r\beta_2}{(1-\beta_2)r}\beta_1^{\frac{r}{\rho-r}}\left[\frac{\beta_1}{1-\beta_2}c(a) + \beta_3\right]^{-\frac{(1-\beta_2)r}{\rho-r}} + A_0\frac{\rho - r\beta_2}{\rho - r}\beta_1^{\frac{r}{\rho-r}}\frac{\beta_1}{1-\beta_2}\left[\frac{\beta_1}{1-\beta_2}c(a) + \beta_3\right]^{-\frac{\rho-r\beta_2}{\rho-r}}\frac{dc(a)}{dA_0}$$

Use the fact that

$$A_0 = \frac{\rho - r}{\rho - r\beta_2}\frac{1-\beta_2}{\beta_1^{\frac{\rho}{\rho-r}}}\left[\frac{\beta_1}{1-\beta_2}(r\underline{a} + y) + \beta_3\right]^{\frac{\rho-r\beta_2}{\rho-r}} > 0$$

since $\frac{\rho-r\beta_2}{1-\beta_2} > 0$, $\rho > r$, and $\beta_3 > -\frac{\beta_1}{1-\beta_2}c(a)$. Inserting for A_0 then gives

$$\frac{dc(a)}{dA_0}[1 - \beta_0] = -\frac{\rho - r\beta_2}{(1-\beta_2)r}\beta_1^{\frac{r}{\rho-r}}\left[\frac{\beta_1}{1-\beta_2}c(a) + \beta_3\right]^{-\frac{(1-\beta_2)r}{\rho-r}} < 0$$

where the inequality follows since $0 < \beta_0 < 1$ from the definition of β_0 in the proof for Theorem 1, $\frac{\rho-r\beta_2}{1-\beta_2} > 0$, $\beta_3 > -\frac{\beta_1}{1-\beta_2}c(a)$, and $\rho > r > 0$ by assumption.

Part II: $\frac{dc_a(a)}{dA_0} > 0$

First, derive $c_a(a)$ by total derivation

$$c_a(a) = \frac{\rho - r\beta_2}{1-\beta_2} + A_0\frac{\rho - r\beta_2}{1-\beta_2}\frac{\beta_1^{\frac{\rho}{\rho-r}}}{\rho - r}\left[\frac{\beta_1}{1-\beta_2}c(a) + \beta_3\right]^{-\frac{\rho-r\beta_2}{\rho-r}}c_a(a)$$

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Apply total derivation to the $c_a(a)$ expression with respect to A_0

$$\frac{dc_a(a)}{dA_0} = \frac{\rho - r\beta_2}{1 - \beta_2}\frac{\beta_1^{\frac{\rho}{\rho-r}}}{\rho - r}\left[\frac{\beta_1}{1 - \beta_2}c(a) + \beta_3\right]^{-\frac{\rho-r\beta_2}{\rho-r}} c_a(a)$$

$$+ A_0\frac{\rho - r\beta_2}{1 - \beta_2}\frac{\beta_1^{\frac{\rho}{\rho-r}}}{\rho - r}\left[\frac{\beta_1}{1 - \beta_2}c(a) + \beta_3\right]^{-\frac{\rho-r\beta_2}{\rho-r}} \frac{dc_a(a)}{dA_0}$$

$$- A_0\frac{\rho - r\beta_2}{1 - \beta_2}\frac{\rho - r\beta_2}{1 - \beta_2}\frac{\beta_1}{\rho - r}\frac{\beta_1^{\frac{\rho}{\rho-r}}}{\rho - r}\left[\frac{\beta_1}{1 - \beta_2}c(a) + \beta_3\right]^{-\frac{\rho-r\beta_2}{\rho-r}-1} c_a(a)\frac{dc(a)}{dA_0}$$

Inserting for A_0 in the second part and moving it to the other side gives

$$\frac{dc_a(a)}{dA_0}[1 - \beta_0] = \frac{\rho - r\beta_2}{1 - \beta_2}\frac{\beta_1^{\frac{\rho}{\rho-r}}}{\rho - r}\left[\frac{\beta_1}{1 - \beta_2}c(a) + \beta_3\right]^{-\frac{\rho-r\beta_2}{\rho-r}} c_a(a)$$

$$- A_0\frac{\rho - r\beta_2}{1 - \beta_2}\frac{\rho - r\beta_2}{1 - \beta_2}\frac{\beta_1}{\rho - r}\frac{\beta_1^{\frac{\rho}{\rho-r}}}{\rho - r}\left[\frac{\beta_1}{1 - \beta_2}c(a) + \beta_3\right]^{-\frac{\rho-r\beta_2}{\rho-r}-1} c_a(a)\frac{dc(a)}{dA_0} > 0$$

where the inequality follows since $\frac{\rho-r\beta_2}{1-\beta_2} > 0$, $\rho > r > 0$, $\beta_3 > -\frac{\beta_1}{1-\beta_2}c(a)$, $c_a(a) > 0$, $A_0 > 0$, and $\frac{dc(a)}{dA_0} < 0$.

Part III: $\frac{dc_{aa}(a)}{dA_0} < 0$

Apply total derivation to the $c_a(a)$ expression with respect to a to obtain the $c_{aa}(a)$ expressions

$$c_{aa}(a) = A_0\frac{\rho - r\beta_2}{1 - \beta_2}\frac{\beta_1^{\frac{\rho}{\rho-r}}}{\rho - r}\left[\frac{\beta_1}{1 - \beta_2}c(a) + \beta_3\right]^{-\frac{\rho-r\beta_2}{\rho-r}} c_{aa}(a) - A_0\left(\frac{\rho - r\beta_2}{1 - \beta_2}\right)^2\frac{\beta_1\beta_1^{\frac{\rho}{\rho-r}}}{(\rho - r)^2}\left[\frac{\beta_1}{1 - \beta_2}c(a) + \beta_3\right]^{-\frac{\rho-r\beta_2}{\rho-r}-1} c_a^2(a) \tag{15}$$

Applying total derivation to the expression with respect to A_0

$$\frac{dc_{aa}(a)}{dA_0} = A_0 \frac{\rho - r\beta_2}{1-\beta_2} \frac{\beta_1^{\frac{\rho}{\rho-r}}}{\rho-r} \left[\frac{\beta_1}{1-\beta_2}c(a) + \beta_3\right]^{-\frac{\rho-r\beta_2}{\rho-r}} \frac{dc_{aa}(a)}{dA_0}$$

$$+ \frac{\rho - r\beta_2}{1-\beta_2} \frac{\beta_1^{\frac{\rho}{\rho-r}}}{\rho-r} \left[\frac{\beta_1}{1-\beta_2}c(a) + \beta_3\right]^{-\frac{\rho-r\beta_2}{\rho-r}} c_{aa}(a)$$

$$- A_0 \frac{\rho - r\beta_2}{1-\beta_2} \frac{\rho - r\beta_2}{1-\beta_2} \frac{\beta_1}{\rho-r} \frac{\beta_1^{\frac{\rho}{\rho-r}}}{\rho-r} \left[\frac{\beta_1}{1-\beta_2}c(a) + \beta_3\right]^{-\frac{\rho-r\beta_2}{\rho-r}} c_{aa}(a)\frac{dc(a)}{dA_0}$$

$$- \frac{\rho - r\beta_2}{1-\beta_2} \frac{\rho - r\beta_2}{1-\beta_2} \frac{\beta_1}{\rho-r} \frac{\beta_1^{\frac{\rho}{\rho-r}}}{\rho-r} \left[\frac{\beta_1}{1-\beta_2}c(a) + \beta_3\right]^{-\frac{\rho-r\beta_2}{\rho-r}-1} c_a^2(a)$$

$$- A_0 \frac{\rho - r\beta_2}{1-\beta_2} \frac{\rho - r\beta_2}{1-\beta_2} \frac{\beta_1}{\rho-r} \frac{\beta_1^{\frac{\rho}{\rho-r}}}{\rho-r} \left[\frac{\beta_1}{1-\beta_2}c(a) + \beta_3\right]^{-\frac{\rho-r\beta_2}{\rho-r}-1} 2c_a(a)\frac{dc_a(a)}{dA_0}$$

$$+ A_0 \frac{\rho - r\beta_2}{1-\beta_2} \frac{\rho - r\beta_2}{1-\beta_2} \frac{\beta_1}{\rho-r} \frac{\beta_1^{\frac{\rho}{\rho-r}}}{\rho-r} \left(\frac{\rho-r\beta_2}{1-\beta_2} + \frac{\rho-r}{1-\beta_2}\right) \frac{\beta_1}{\rho-r} \left[\frac{\beta_1}{1-\beta_2}c(a) + \beta_3\right]^{-\frac{\rho-r\beta_2}{\rho-r}-2} c_a^2(a)\frac{dc(a)}{dA_0}$$

Replacing for A_0 in the first expression gives

$$\frac{dc_{aa}(a)}{dA_0}[1 - \beta_0] = \frac{\rho - r\beta_2}{1-\beta_2} \frac{\beta_1^{\frac{\rho}{\rho-r}}}{\rho-r} \left[\frac{\beta_1}{1-\beta_2}c(a) + \beta_3\right]^{-\frac{\rho-r\beta_2}{\rho-r}} c_{aa}(a)$$

$$- A_0 \frac{\rho - r\beta_2}{1-\beta_2} \frac{\rho - r\beta_2}{1-\beta_2} \frac{\beta_1}{\rho-r} \frac{\beta_1^{\frac{\rho}{\rho-r}}}{\rho-r} \left[\frac{\beta_1}{1-\beta_2}c(a) + \beta_3\right]^{-\frac{\rho-r\beta_2}{\rho-r}} c_{aa}(a)\frac{dc(a)}{dA_0}$$

$$- \frac{\rho - r\beta_2}{1-\beta_2} \frac{\rho - r\beta_2}{1-\beta_2} \frac{\beta_1}{\rho-r} \frac{\beta_1^{\frac{\rho}{\rho-r}}}{\rho-r} \left[\frac{\beta_1}{1-\beta_2}c(a) + \beta_3\right]^{-\frac{\rho-r\beta_2}{\rho-r}-1} c_a^2(a)$$

$$- A_0 \frac{\rho - r\beta_2}{1-\beta_2} \frac{\rho - r\beta_2}{1-\beta_2} \frac{\beta_1}{\rho-r} \frac{\beta_1^{\frac{\rho}{\rho-r}}}{\rho-r} \left[\frac{\beta_1}{1-\beta_2}c(a) + \beta_3\right]^{-\frac{\rho-r\beta_2}{\rho-r}-1} 2c_a(a)\frac{dc_a(a)}{dA_0}$$

$$+ A_0 \frac{\rho - r\beta_2}{1-\beta_2} \frac{\rho - r\beta_2}{1-\beta_2} \frac{\beta_1}{\rho-r} \frac{\beta_1^{\frac{\rho}{\rho-r}}}{\rho-r} \frac{\beta_1}{\rho-r} \left(\frac{\rho-r\beta_2}{1-\beta_2} + \frac{\rho-r}{1-\beta_2}\right) \left[\frac{\beta_1}{1-\beta_2}c(a) + \beta_3\right]^{-\frac{\rho-r\beta_2}{\rho-r}-2} c_a^2(a)\frac{dc(a)}{dA_0} < 0$$

where the last inequality follows because $0 < \beta_0 < 1$, $c_{aa}(a) < 0$ (from Theorem 1), $\frac{dc(a)}{dA_0} < 0$, $c_a(a) > 0$ (from Theorem 1), $\frac{dc_a(a)}{dA_0} > 0$, $\rho > r > 0$, $\beta_1 > 0$, $\frac{\rho-r\beta_2}{1-\beta_2} > 0$, $\frac{\rho-r\beta_2}{1-\beta_2} + \frac{\rho-r}{1-\beta_2} > 0$, and $\beta_3 > -\frac{\beta_1}{1-\beta_2}c(a)$. Furthermore, since κ is a monotone increasing function of A_0 by Lemma 4, all the results above also holds for κ. $\qquad\square$