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by

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Simulating forward-looking models

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Abstract

The solutions of a macroeconometric model with expectations of future-dated variables has to be approximated by numerical simulations. A brief review of deterministic and stochastic dynamic simulations of a backward-looking model is followed by a *conceptual* presentation of methods for dynamic simulations of a forward-looking (rational expectations) model. Detailed numerical methods for solving the models are beyond the scope of this note.

Keywords: *Forward-looking models, rational expectations, simultaneous equation systems, stochastic simulation*

JEL classifications: C15, C30, C63

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1 Introduction

Solutions of a dynamic macroeconometric model with expectations of future-dated variables (called leads) have to be approximated numerically by simulation. There is more than one way to proceed, and we present different methods and sketch different algorithms. In particular we seek to clarify some complicating issues that have to be dealt with when undertaking *stochastic simulations* in a rational expectations model with leading variables.

There are many sources of uncertainty in an econometric model. Fair (1984) distinguishes between stochastic disturbances/shocks, estimated rather than known parameter values, exogenous variables and the simplified structure of the model. Gallo and Don (1991) add uncertainty due to unreliable data. We limit our discussion to stochastic simulation of a given model, where shocks and estimates of unknown parameters are the only recognized sources of uncertainty. We condition on the model and the exogenous values. In light of the partial incorporation of uncertainty in the simulations, it seems reasonable to view the simulated variability as a conservative estimate of the predictive uncertainty. As the paper focuses on simulation techniques we do not pursue this discussion any further. Neither modelling nor estimation are issues of this paper.

When doing stochastic simulation of a model with rational agents we have to distinguish between uncertainty faced by the modeller (imprecise estimates and unexplained shocks) and uncertainty faced by rational expectation forming agents (unforseeable shocks). For the simulation disturbances we discuss three alternatives: Agents have perfect foresight, know the distribution of the shocks or only the expectations. As for the parameter estimates we have to distinguish between knowledge of the modeller and the rational agents. Unfortunately, implementing and testing the algorithms on an operative macroeconometric model are beyond the scope of this paper. Nevertheless, this note might serve as a preparatory introduction to such an undertaking.

The structure of this paper is as follows. The next section reviews the backward-looking model, while section 3 looks at solutions of the nonlinear model by stochastic simulations. These two sections serve as background for the following sections and their methods. The forward-looking model is discussed in section 4, which presents the iterative method of Fair and Taylor (1983) and the more recent stacked-time method, see e.g. Armstrong et al. (1998), Hollinger (1996), Juillard et al. (1998). The fifth section continues with stochastic simulation of the forward-looking model. Different algorithms are sketched, first with stochastic disturbances only, and then with stochastic parameter estimates in addition. A final section concludes.

2 Simulation of a backward-looking model

A macroeconometric model is a nonlinear dynamic simultaneous equation system. Without forward-looking behaviour its structural equations can be written generally as

$$\mathbf{f}(\mathbf{y}_t, \mathbf{y}_{t-1}, \mathbf{x}_t, \beta) = \mathbf{u}_t \sim \mathsf{IID}(\mathbf{0}, \mathbf{\Sigma}), \tag{1}$$

where $\mathbf{f}$ denotes a vector of functions and lag operators. The arguments $\mathbf{y}_t$, $\mathbf{y}_{t-1}$, $\mathbf{x}_t$ and β are vectors of current and lagged endogenous variables, exogenous variables and structural

parameters (coefficients). The vectors of structural errors or shocks, $\mathbf{u}_t$, are assumed to be intertemporally independent and identically distributed (IID), with zero mean vector and a contemporaneous covariance matrix $\mathbf{\Sigma}$. The dynamic structure of the model is made explicit by the inclusion of a single lag $\mathbf{y}_{t-1}$, which is fully general[1]. Lags in the exogenous variables are irrelevant for the discussion, and are dropped to simplify notation. Definitional equations do not contribute to the simulation properties of the model and are consequently ignored.

The parameters are estimated from T observations on $\mathbf{x}_t$ and (due to the lag) $T+1$ observations on $\mathbf{y}_t$. The estimated model is

$$\mathbf{f}(\mathbf{y}_t, \mathbf{y}_{t-1}, \mathbf{x}_t, \hat{\beta}) = \hat{\mathbf{u}}_t \sim \mathsf{ID}(\mathbf{0}, \widehat{\mathbf{\Sigma}}), \quad t = 1, \ldots, T, \tag{2}$$

Hats denote estimated values. The vector $\hat{\beta}$ contains the estimated parameter values. By construction the vectors of empirical residuals $\hat{\mathbf{u}}_t$ are dependent, but approximately identically distributed (ID) and serially uncorrelated, with zero mean and an empirical covariance matrix $\widehat{\mathbf{\Sigma}}$. We have used the terms *error* or *shocks* for the stochastic variables $\mathbf{u}_t$ and *residual* for their empirical values $\hat{\mathbf{u}}_t$.

The system of nonlinear equations (1) generally does not yield a closed form solution for $\mathbf{y}_t$ in terms of the predetermined variables $\mathbf{y}_{t-1}, \mathbf{x}_t, \mathbf{u}_t$ and the parameters β. However, for relevant values of the variables the system implicitly defines a presumably unique solution, expressed by $\mathbf{y}_t = \mathbf{g}(\mathbf{y}_{t-1}, \mathbf{x}_t, \mathbf{u}_t, \beta)$. The standard numerical approximation to this solution is a deterministic dynamic simulation of the estimated model (2),

$$\tilde{\mathbf{y}}_t = \mathbf{g}(\tilde{\mathbf{y}}_{t-1}, \mathbf{x}_t, \tilde{\mathbf{u}}_t, \hat{\beta}), \quad \tilde{\mathbf{y}}_0 = \mathbf{y}_0, \quad t = 1, \ldots, T, \tag{3}$$

where tildes denote simulated values. The simulation disturbances are denoted by $\tilde{\mathbf{u}}_t$, which are any input values representing the error terms or shocks. When $\hat{\mathbf{u}}_t$ is put into an in-sample simulation the estimated model (2) reproduces data exactly, and $\tilde{\mathbf{y}}_t = \mathbf{y}_t$ for $t = 1, \ldots, T$. Common procedure is to input either the expected values of the shocks, $\tilde{\mathbf{u}}_t = \mathbf{0}$ or, in an in-sample simulation, the empirical residuals, $\tilde{\mathbf{u}}_t = \hat{\mathbf{u}}_t$ as intercept corrections. We let such intercept corrections be represented by exogenous variables.

When the model is linear in variables (1) can be written as

$$\mathbf{A}\mathbf{y}_t - \mathbf{B}\mathbf{y}_{t-1} - \mathbf{C}\mathbf{x}_t = \mathbf{u}_t,$$

where $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are coefficient matrices. Constant terms can be represented by constants in $\mathbf{C}$ and $\mathbf{x}$. For an invertible $\mathbf{A}$

$$\mathbf{y}_t = \mathbf{A}^{-1}\mathbf{B}\mathbf{y}_{t-1} + \mathbf{A}^{-1}\mathbf{C}\mathbf{x}_t + \mathbf{A}^{-1}\mathbf{u}_t.$$

The endogenous variables are functions of the error terms. Consequently they are stochastic variables, and

$$\mathsf{E}[\mathbf{y}_t \mid \mathbf{x}_t] = \mathbf{A}^{-1}\mathbf{B}\mathsf{E}\mathbf{y}_{t-1} + \mathbf{A}^{-1}\mathbf{C}\mathbf{x}_t$$

[1]For a linear (in parameters) model this can be seen writing the model in companion form, cf. Hendry (1995, ch. 8 and A1.6). This idea of adding current and lagged variables defined as higher order lags of existing variables carries over to a nonlinear model. Alternatively, the vector $\mathbf{f}$ of functions and/or lag operators can contain higher order lag operators.

as $\mathsf{E}[\mathbf{u}_t \mid \mathbf{x}_t] = \mathbf{0}$. With zero simulation disturbances the linear version of (3),

$$\tilde{\mathbf{y}}_t = \hat{\mathbf{A}}^{-1}\hat{\mathbf{B}}\tilde{\mathbf{y}}_{t-1} + \hat{\mathbf{A}}^{-1}\hat{\mathbf{C}}\mathbf{x}_t \equiv \hat{\mathbf{\Pi}}\,\tilde{\mathbf{y}}_{t-1} + \hat{\mathbf{\Gamma}}\,\mathbf{x}_t,$$

approximates the conditional expected values of the endogenous variables, $\tilde{\mathbf{y}}_t \approx \mathsf{E}[\mathbf{y}_t \mid \mathbf{x}_t]$. The approximate equality results from the fact that only estimates of the true values of the parameters are known. When expectation values of stochastic input variables are mapped onto expectation values of stochastic output values we speak of certainty-equivalence. For linear models the property of certainty-equivalence allows the model operator to find the expected values of the endogenous variables in one single deterministic simulation. For nonlinear models certainty-equivalence does not hold, as in general

$$\mathsf{E}[\mathbf{y}_t \mid \mathbf{x}_t] = \mathsf{E}\mathbf{g}(\tilde{\mathbf{y}}_{t-1}, \mathbf{x}_t, \tilde{\mathbf{u}}_t, \hat{\beta}) \neq \mathbf{g}(\mathsf{E}\mathbf{y}_{t-1}, \mathbf{x}_t, \mathbf{0}, \hat{\beta}).$$

If the model is only mildly nonlinear, say including slightly curved functions of endogenous variables in only a few equations, then the zero residual deterministic simulation might still be a reasonable approximation to the expected values of the endogenous variables, see Fair (1984), Fisher and Salmon (1986), Hall and Henry (1988). The common methodology of deterministic simulation implicitly follows the standard statistical practice of plugging in the best estimate of unknown model parameters, and treating as stochastic only variables specified to be stochastic. Below we shall relax this practice by considering the parameter *estimates* as stochastic too.

3 Stochastic simulation of a backward-looking model

From the specification and estimation of the econometric model it follows that the shocks $\mathbf{u}_t$ in (1) and the estimates $\hat{\beta}$ in (2) are stochastic variables. The premise of parameter constancy is not violated, since it is the *estimator* that is stochastic. The unknown parameters β are still perceived as constants. The only model input being deterministic, by definition, is the exogenous variables $\mathbf{x}$. In ex post (within sample) simulations their values are historical observations or counterfactual values upon which the simulations, and consequently the conclusions, are conditioned. When simulating the estimated model (2) one can take into account these sources of variability which lead to uncertainty in the model solutions.

The mapping of variables implies a mapping of probability distributions of stochastic input variables onto a distribution of stochastic output variables. Classical econometrics let us assume that the stochastic input variables are independent. Then the two mappings can be schematically expressed as

$$\begin{array}{rcl}
 & & \hat{\beta} \\
 & & \downarrow \\
\tilde{\mathbf{y}}_{t-1}, \mathbf{x}_t, \tilde{\mathbf{u}}_t & \rightarrow & \boxed{\text{model}} \rightarrow \tilde{\mathbf{y}}_t \\
 & \Downarrow & \\
\mathsf{D}_{\tilde{\mathbf{y}},t-1}(\tilde{\mathbf{y}}_{t-1}),\ \mathbf{x}_t,\ \mathsf{D}_{\tilde{\mathbf{u}}}(\tilde{\mathbf{u}}_t) & \rightarrow & \boxed{\text{model}} \rightarrow \mathsf{D}_{\tilde{\mathbf{y}},t}(\tilde{\mathbf{y}}_t), \\
 & & \uparrow \\
 & & \mathsf{D}_{\hat{\beta}}(\hat{\beta})
\end{array}$$

where the Ds denote the joint distributions of the simulated endogenous variables, the joint distributions of the simulation disturbances and the joint distributions of the parameter estimates. A time subscript on the distribution of $\mathbf{y}$ accommodates nonstationarity. Since the mapping of the input variables is known only implicitly through the structural model (2), the resulting mapping of the variables' distributions is unknown. But, the unknown output distributions $\mathsf{D}_{\tilde{\mathbf{y}},t}(\tilde{\mathbf{y}}_t)$, $t = 1, \ldots T$, can be estimated by dynamic stochastic simulations. The central limit theorem suggests that for weakly nonlinear models we may anticipate approximate normality of the endogenous variables[2].

The term deterministic simulation is synonymous with an iterative numerical solution of the deterministic model. The term stochastic simulation covers an additional operation. First sampling the stochastic input variables, then approximating the implicit reduced form solution by numerical iteration. The two tasks can be seen as first sampling on the input side directly and then on the output side of the model indirectly. Stochastic simulation is basically about sampling the distributions of the stochastic variables of the estimated model. The sampling is carried out simply by replicating a single ("deterministic") simulation $n = 1, \ldots, N$ times. Each replication n is simulated with new values of the stochastic input variables randomly drawn from their respective postulated and estimated input distributions. A very simple way to do this is to assume multivariate normal distributions for the error terms and, consequently, for the parameter estimates, and to center the distributions on the empirical residuals (to be discussed) and the estimated values. Denoting a simulated random variable by a tilde and a replication superscript $^{(n)}$ (while a hat denotes an estimate) stochastic simulations are performed according to:

Algorithm 1: Stochastic simulation of a backward-looking model

For $n = 1, 2, \ldots, N$ replications of a single dynamic simulation

$\tilde{\beta}^{(n)} \sim \mathsf{I}_n\mathsf{IN}(\hat{\beta}, \widehat{\mathbf{\Omega}})$,

For $t = 1, 2, \ldots, T$ periods within a single dynamic simulation

$\tilde{\mathbf{u}}_t^{(n)} \sim \mathsf{I}_{n,t}\mathsf{IN}(\mathbf{0}, \widehat{\mathbf{\Sigma}})$,

$\tilde{\mathbf{y}}_t^{(n)} = \mathbf{g}(\tilde{\mathbf{y}}_{t-1}^{(n)}, \mathbf{x}_t, \tilde{\mathbf{u}}_t^{(n)}, \tilde{\beta}^{(n)}), \quad \tilde{\mathbf{y}}_0^{(n)} = \mathbf{y}_0$.

The vector of simulation parameters $\tilde{\beta}^{(n)}$ is independent over replications, denoted by I_n, and identically distributed normal (IN), centered on the estimate with empirical covariance matrix $\widehat{\mathbf{\Omega}}$. Its value only changes with the replication n, and remains constant through the T simulation periods. The vector of simulation disturbances $\tilde{\mathbf{u}}_t$ (which is not strictly the error term in the model (1)) is independent over both replications and time, denoted by $\mathsf{I}_{n,t}$, and identically distributed normal, centered on the expectation $\mathbf{0}$. Its covariance structure is that of the empirical residuals. A new value is drawn every simulation period t in every replication n. Stochastic perturbations to realized or expectation values of the error term, and to the estimates provide a simple and common way to generate stochastic variation, see

[2]The Lindeberg generalization of the central limit theorem implies under general conditions (applicable in nearly every practical situation) that a sum of a large number of independent random variables with finite moments converges to a normal distribution, see Davidson and McKinnon (1993, ch. 4.7).

Fair (1984) and Hollinger (1990). A single multivariate path (multiple time series) over $t = 1, \ldots, T$ periods is simulated in each replication $n = 1, \ldots, N$. The path is a hypothetically possible realization of the model economy. The stochastic simulation procedure given by the algorithm above yields a bundle of different paths $\{(y_1^{(n)}, y_2^{(n)}, \ldots, y_T^{(n)})\}_{n=1}^N$The dispersion of the sample paths reflects conditional uncertainty in the model solutions.

4 Simulation of a forward-looking model

When the model (1) in addition to lagged variables contains expectations of future-dated variables (leads) $\mathbf{y}_{t+1}$ it can be modified to

$$\mathbf{f}(\mathbf{y}_t, \mathbf{y}_{t-1}, \mathsf{E}[\mathbf{y}_{t+1} \mid \mathcal{I}_{t-1}], \mathbf{x}_t, \beta) = \mathbf{u}_t \sim \mathsf{IID}(\mathbf{0}, \mathbf{\Sigma}), \tag{4}$$

where $\mathcal{I}_{t-1} \subseteq \{\mathbf{y}_{t-1}, \beta, \mathsf{E}\mathbf{u}_t = \mathbf{0}, \mathbf{\Sigma}, \mathsf{E}_{t-1}\mathbf{x}_t = \mathbf{x}_t, \mathsf{E}_{t-1}\mathbf{u}_t = \mathbf{0} \vee \mathbf{u}_t\}$ denotes the information set at the end of period $t-1$. As with the lags the inclusion of a single lead is fully general. With the shorthand notation $\bar{\mathbf{y}}_{t+1} = \mathsf{E}[\mathbf{y}_{t+1} \mid \mathcal{I}_{t-1}]$ we have the forward-looking version of the implicit solution (3)

$$\tilde{\mathbf{y}}_t = \mathbf{g}(\tilde{\mathbf{y}}_{t-1}, \bar{\mathbf{y}}_{t+1}, \mathbf{x}_t, \tilde{\mathbf{u}}_t, \hat{\beta}), \quad \tilde{\mathbf{y}}_0 = \mathbf{y}_0, \quad t = 1, \ldots, T. \tag{5}$$

While the backward-looking model (2) could be solved recursively by numerical iteration to yield the implicit solution (3), the solution (5) of the forward-looking model depends on expectations of the future in addition to realizations of the past. This has severe implications on both the uniqueness of the solution and the method of approximating one.

When the model is linear in variables (4) can be written as

$$\mathbf{A}\mathbf{y}_t + \mathbf{B}\mathbf{y}_{t-1} + \mathbf{C}\mathbf{x}_t - \mathbf{D}\mathsf{E}[\mathbf{y}_{t+1} \mid \mathcal{I}_{t-1}] = \mathbf{u}_t,$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are coefficient matrices. Constant terms can be represented by constants in $\mathbf{C}$ and $\mathbf{x}$. When $\mathbf{A}$ is invertible the linear version of (5) becomes

$$\begin{aligned} \tilde{\mathbf{y}}_t &= -\hat{\mathbf{A}}^{-1}\hat{\mathbf{B}}\tilde{\mathbf{y}}_{t-1} - \hat{\mathbf{A}}^{-1}\hat{\mathbf{C}}\mathbf{x}_t + \hat{\mathbf{A}}^{-1}\hat{\mathbf{D}}\tilde{\mathbf{y}}_{t+1} \\ &\equiv \hat{\mathbf{\Pi}}\tilde{\mathbf{y}}_{t-1} + \hat{\mathbf{\Gamma}}\mathbf{x}_t + \hat{\mathbf{\Upsilon}}\tilde{\mathbf{y}}_{t+1} \end{aligned} \tag{6}$$

by certainty-equivalence: $\mathsf{E}[\mathbf{u}_t \mid \mathcal{I}_{t-1}] = \mathbf{0} \Rightarrow \bar{\mathbf{y}}_{t+1} = \mathsf{E}[\mathbf{y}_{t+1} \mid \mathcal{I}_{t-1}] = \tilde{\mathbf{y}}_{t+1}$. Unlike the linear backward-looking model the solutions cannot be computed recursively since the current solution depends on future solutions in addition to past solutions. Generally, the solutions for all periods have to be found simultaneously.

The model (6) can alternatively be written with $\tilde{\mathbf{y}}_{t+1}$ as the left hand side variable:

$$\hat{\mathbf{D}}\tilde{\mathbf{y}}_{t+1} = \hat{\mathbf{A}}\tilde{\mathbf{y}}_t + \hat{\mathbf{B}}\tilde{\mathbf{y}}_{t-1} + \hat{\mathbf{C}}\mathbf{x}_t$$

Then, *if* $\mathbf{D}$ had full rank the recursion

$$\begin{aligned} \tilde{\mathbf{y}}_{t+1} &= \hat{\mathbf{D}}^{-1}\hat{\mathbf{A}}\tilde{\mathbf{y}}_t + \hat{\mathbf{D}}^{-1}\hat{\mathbf{B}}\tilde{\mathbf{y}}_{t-1} + \hat{\mathbf{D}}^{-1}\hat{\mathbf{C}}\mathbf{x}_t \\ &\equiv \hat{\mathbf{\Psi}}\tilde{\mathbf{y}}_t + \hat{\mathbf{\Pi}}\tilde{\mathbf{y}}_{t-1} + \hat{\mathbf{\Gamma}}\mathbf{x}_t. \end{aligned}$$

would approximate the expected values of the endogenous variables. But, since the model is far from likely to contain expectations of all endogenous variables, the coefficient matrix $\mathbf{D}$ almost surely has reduced rank. Then it cannot be inverted, and the solution for all datings of the endogenous variables have to be found simultaneously rather than recursively. This shows that expectations of future-dated variables complicate the solution procedure for linear as well as nonlinear models. There is a relatively large literature on linear rational expectations models compared to nonlinear models, see for instance Gourieroux and Monfort (1997, ch. 12), Holly and Hughes-Hallett (1989, ch. 7) and Wallis (1980). Fisher (1992) presents a comprehensive set of techniques for handling nonlinear as well as linear econometric models with rational expectations.

There are several different methods of finding a solution (5) for the general nonlinear forward-looking model (4). If the expectation values $\bar{\mathbf{y}}_{t+1}$ were known, along with the already simulated values of the lagged endogenous variables $\tilde{\mathbf{y}}_{t-1}$, the exogenous values $\mathbf{x}_t$ and the simulation disturbances $\tilde{\mathbf{u}}_t$, the solution $\tilde{\mathbf{y}}_t$ could be found by the same numerical simulation procedure as for the backward-looking model (2). The two solution methods to be reviewed actually use the same numerical simulation procedure as for the backward-looking model. They differ in how they treat the expectations $\bar{\mathbf{y}}_{t+1}$ to be able to do that. The perhaps most well known method is the iterative method due to Fair and Taylor (1983, 1990), denoted FT. Another more recent method (available in the TROLL software) is the Stacked-Time method, denoted ST, see e.g. Fisher et al. (1986), Hollinger (1996). To decide the terminal conditions, that is the values for the expectations beyond the last simulation period, $\bar{\mathbf{y}}_{T+1}$, FT and ST both use the extended path method. The two methods are not disjunct. In the TROLL software ST can be a part of the FT-method. The next two subsections explain why this is so, as do Armstrong et al. (1998).

4.1 The Fair-Taylor method (FT)

This method starts by choosing an initial time path for the expectations, extended k periods beyond the original interval:

$$\bar{\mathbf{Y}}^1 = (\underbrace{\bar{\mathbf{y}}^1_2, \ldots, \bar{\mathbf{y}}^1_{T+1}}_{\text{original interval}}, \underbrace{\bar{\mathbf{y}}^1_{T+2}, \ldots, \bar{\mathbf{y}}^1_{T+k+1}}_{\text{extension}}),$$

where $k \geq 1$. The length of the time extension $\bar{\mathbf{y}}^1_{T+2}, \ldots, \bar{\mathbf{y}}^1_{T+k+1}$, that is the size of k, depends on what one assumes or knows about the values in the final period T. The reason for the extension will be clear shortly. Given these values of the expectation variables the model can be solved by recursive dynamic simulation, just like a backward-looking model. The solution of the model given the expectations $\bar{\mathbf{Y}}^1$ is the time path

$$\tilde{\mathbf{Y}}^1 = (\underbrace{\tilde{\mathbf{y}}^1_1, \ldots, \tilde{\mathbf{y}}^1_{T+1}}_{\text{original interval}}, \underbrace{\tilde{\mathbf{y}}^1_{T+2}, \ldots, \tilde{\mathbf{y}}^1_{T+k}}_{\text{extension}}, \underbrace{\bar{\mathbf{y}}^1_{T+k+1}}_{\text{terminal value}}),$$

where

$$\tilde{\mathbf{y}}^1_t = \mathbf{g}(\tilde{\mathbf{y}}^1_{t-1}, \bar{\mathbf{y}}^1_{t+1}, \mathbf{x}_t, \tilde{\mathbf{u}}_t, \hat{\beta}), \quad \tilde{\mathbf{y}}^1_0 = \mathbf{y}_0, \quad t = 1, \ldots, T+k. \tag{7}$$

For each period t the solution (7) is approximated by iteration. Fair and Taylor (1983) denote these iterations required for each period-specific solution Type I iterations. The FT-method proceeds to iterate on the solution path (of Type I solutions) by letting the simulated values replace the expectation values, setting $\bar{\mathbf{Y}}^{i+1} = \tilde{\mathbf{Y}}^i$, $i = 1, 2, \ldots$, that is $\bar{\mathbf{y}}_t^{i+1} = \tilde{\mathbf{y}}_t^i$ for $t = 2, \ldots, T+k$, before resimulating the whole time span $t = 1, 2, \ldots, T+k$.

The method presupposes that each new dynamic simulation $\tilde{\mathbf{Y}}^i$ is closer to the true solution than the previous one $\tilde{\mathbf{Y}}^{i-1}$, since each simulation then uses an improved approximation to the true expectation values. When $\left|\tilde{\mathbf{y}}_t^i - \tilde{\mathbf{y}}_t^{i-1}\right| \leq \delta_{II}$ for every $t = 1, \ldots, T+1$, where δ_{II} is a prescribed tolerance level, convergence *on the original interval* has been achieved after i iterations. Each of the i iterations is called a Type II iteration. While Type I iterations are for each solution period, each Type II iteration is for the whole solution path.

We denote the Type II solution path $\tilde{\mathbf{Y}}^i$ by $\tilde{\mathbf{Y}}$ only. It depends on the terminal expectation values $\bar{\mathbf{y}}_{T+k+1}^1$ and the length $k-1$ of the extension. We denote this by

$$\begin{aligned}
\tilde{\mathbf{Y}}(k, \bar{\mathbf{y}}_{T+k+1}^1) &= (\underbrace{\tilde{\mathbf{y}}_1, \ldots, \tilde{\mathbf{y}}_{T+1}}_{\text{original interval}}, \underbrace{\tilde{\mathbf{y}}_{T+2}, \ldots, \tilde{\mathbf{y}}_{T+k}}_{\text{extension}}, \underbrace{\bar{\mathbf{y}}_{T+k+1}^1}_{\text{terminal value}}) \\
&= (\tilde{\mathbf{y}}_1, \ldots, \tilde{\mathbf{y}}_{T+1}) \vdash (\tilde{\mathbf{y}}_{T+2}, \ldots, \tilde{\mathbf{y}}_{T+k}) \vdash \bar{\mathbf{y}}_{T+k+1}^1 \\
&\equiv \tilde{\mathbf{Y}}_{org}(k, \bar{\mathbf{y}}_{T+k+1}^1) \vdash \tilde{\mathbf{Y}}_{ext}(k, \bar{\mathbf{y}}_{T+k+1}^1) \vdash \bar{\mathbf{y}}_{T+k+1}^1,
\end{aligned}$$

where $\vdash$ is a concatenation operator. The terminal vector $\bar{\mathbf{y}}_{T+k+1}^1$ has not been changed during the simulations since its dating is outside the simulation time span.

We only need the solution path $\tilde{\mathbf{Y}}_{org}(k, \bar{\mathbf{y}}_{T+k+1}^1)$ through the $T+1$ periods of the original interval. We want this path to be independent of the choice of the terminal vector $\bar{\mathbf{y}}_{T+k+1}^1$. This can be achieved by pushing it sufficiently far into the future. That is the reason for the (length k of the) extension. Whether the subpath $\tilde{\mathbf{Y}}_{org}(k, \bar{\mathbf{y}}_{T+k+1}^1)$ is approximately independent of the terminal vector can be checked by adding another period $T+k+2$. A new terminal vector $\bar{\mathbf{y}}_{T+k+2}^1$ is chosen, and the initial expectation path $\bar{\mathbf{Y}}^1 = \tilde{\mathbf{Y}}(k, \bar{\mathbf{y}}_{T+k+1}^1) \vdash \bar{\mathbf{y}}_{T+k+2}^{k+1,1}$ formed. Type II iterations are again performed until convergence, yielding

$$\begin{aligned}
\tilde{\mathbf{Y}}(k+1, \bar{\mathbf{y}}_{T+k+2}^1) &= (\underbrace{\tilde{\mathbf{y}}_1, \ldots, \tilde{\mathbf{y}}_{T+1}}_{\text{original interval}}, \underbrace{\tilde{\mathbf{y}}_{T+2}, \ldots, \tilde{\mathbf{y}}_{T+k}, \tilde{\mathbf{y}}_{T+k+1}}_{\text{extension}}, \underbrace{\bar{\mathbf{y}}_{T+k+2}^1}_{\text{terminal value}}) \\
&\equiv \tilde{\mathbf{Y}}_{org}(k+1, \bar{\mathbf{y}}_{T+k+2}^1) \vdash \tilde{\mathbf{Y}}_{ext}(k+1, \bar{\mathbf{y}}_{T+k+2}^1) \vdash \bar{\mathbf{y}}_{T+k+2}^1.
\end{aligned}$$

Now, if $\max_t \left|\tilde{\mathbf{Y}}_{org}(k+1, \bar{\mathbf{y}}_{T+k+2}^1) - \tilde{\mathbf{Y}}_{org}(k, \bar{\mathbf{y}}_{T+k+1}^1)\right| \leq \delta_{III}$, where δ_{III} is another prescribed tolerance level, the solution path through the original simulation interval, $\tilde{\mathbf{Y}}_{org} = (\tilde{\mathbf{y}}_1, \ldots, \tilde{\mathbf{y}}_{T+1})$ is practically independent of the length of the extension and the terminal vector. We then have a Type III solution path $\tilde{\mathbf{Y}}_{org}$ after only one Type III iteration. If the convergence criteria is not met, FT iterates on the full solution path $\tilde{\mathbf{Y}}(k, \bar{\mathbf{y}}_{T+k+1}^1)$ by extending it one period ($k \leftarrow k+1$) each Type III iteration. When $\tilde{\mathbf{Y}}_{org}(k, \bar{\mathbf{y}}_{T+k+1}^1) \approx \tilde{\mathbf{Y}}_{org}(j, \bar{\mathbf{y}}_{T+j+1}^1)$ for all $j > k$ irrespective of $\bar{\mathbf{y}}_{T+j+1}^1$ we have a Type III- and final solution. Each Type III iteration involves several Type II iterations, which again performs Type I iterations for each period.

So far we have not mentioned the disturbances nor the exogenous input. The exogenous values are either in-sample observations, with a possibility of counterfactual changes, or projected future values. For in-sample simulations the empirical residuals $\hat{\mathbf{u}}_t$ are often used as intercept corrections. We let such intercept corrections be represented by exogenous variables ($x_{j,t} \in \mathbf{x}_t$). Assuming approximate certainty-equivalence, $\tilde{\mathbf{u}}_t = \mathsf{E}[\mathbf{u}_t \mid \mathcal{I}_{t-1}] = \mathbf{0}$ implies $\bar{\mathbf{y}}_{t+1} \approx \tilde{\mathbf{y}}_{t+1}$, and (5) becomes

$$\tilde{\mathbf{y}}_t = \mathbf{g}(\tilde{\mathbf{y}}_{t-1}, \tilde{\mathbf{y}}_{t+1}, \mathbf{x}_t, \mathbf{0}, \hat{\beta}), \quad \tilde{\mathbf{y}}_0 = \mathbf{y}_0, \quad t = 1, \ldots, T. \tag{8}$$

This is the solution Fair and Taylor approximate by the iterative technique outlined above, (discarding periods $T+1, \ldots, T+k$). The structure of the Fair-Taylor algorithm is

Initialize
Set start values and choose expectation paths through all periods including the terminal,
Type III iterations
Do until Type III solutions on *the original simulation interval* do not change significantly between Type III iterations:
  Extend the simulation path with a new terminal period,
  Type II iterations (for the given extension)
  Do until Type II solutions for *all periods* do not change significantly between Type II iterations:
    Set expectations equal to previous Type II solutions,
    Type I iterations (for each simulation period)
    With expectations equal to the previous (Type II) solutions
    simulate through all periods but the terminal,
  End Do-loop for Type II iterations
  Let the new Type III solutions be equal to the final Type II solutions,
End Do-loop for Type III iterations

To simplify notation in the algorithms, where it is not necessary we do not distinguish notationally between expectations $\bar{\mathbf{y}}_t$ and simulations $\tilde{\mathbf{y}}_t$ but use only $\mathbf{y}_t$. The actual algorithm is

Algorithm 2: Fair-Taylor deterministic simulation of a forward-looking model
Initialize
Prescribe the tolerance levels δ_{II} and δ_{III},
Decide the initial extension length l, and let $k = l - 1$,
Choose an extended start path $\mathbf{Y}(k, \mathbf{y}_{T+k+1}) = (\mathbf{y}_t^1)_{t=1}^{T+k+1}$,
Extend the exogenous path with $\mathbf{x}_{T+1}, \ldots, \mathbf{x}_{T+k}$,

$\Delta_{III} = \infty$,

Type III iterations (over k)

Do until $\Delta_{III} \leq \delta_{III}$

$k = k + 1$ and $i = 1$,

Choose vectors for the added final period: $\mathbf{y}^1_{T+k+1}$ and $\mathbf{x}_{T+k}$,

$\mathbf{Y}^1(k, \mathbf{y}^1_{T+k+1}) = \mathbf{Y}(k-1, \mathbf{y}^1_{T+k}) \vdash \mathbf{y}^1_{T+k+1}$

$\Delta_{II} = \infty$,

Type II iterations (over i)

Do until $\Delta_{II} \leq \delta_{II}$

$i = i + 1$,

Type I iterations (for each period t)

Simulate $\mathbf{Y}^i(k, \mathbf{y}^1_{T+k+1})$ with $\mathbf{Y}^{i-1}(k, \mathbf{y}^1_{T+k+1})$ as expectations

End For-*loop with Type I iterations,*

Let $\Delta_{II} = \max_t \left|\mathbf{Y}^i(k, \mathbf{y}^1_{T+k+1}) - \mathbf{Y}^{i-1}(k, \mathbf{y}^1_{T+k+1})\right|$

End Do-*loop for Type II iterations,*

Let $\mathbf{Y}(k, \mathbf{y}^1_{T+k+1}) = \mathbf{Y}^i(k, \mathbf{y}^1_{T+k+1})$,

Let $\Delta_{III} = \max_t \left|\mathbf{Y}_{org}(k, \mathbf{y}^1_{T+k+1}) - \mathbf{Y}_{org}(k-1, \mathbf{y}^1_{T+k})\right|$

End Do-*loop for Type III iterations*

The backward-looking model in the previous sections is simultaneous across equations within the current period. The forward-looking model with model-consistent leads (rational expectations) is simultaneous across time as well as across equations. The Type I iterations solve the simultaneity of current endogenous variables, while the Type II and Type III iterations solve the simultaneity of current and future periods.

4.2 The Stacked-Time method (ST)

Like FT the ST-method approximates solutions of $\mathbf{f}(\mathbf{y}_t, \mathbf{y}_{t-1}, \mathbf{y}_{t+1}, \mathbf{x}_t, \hat{\beta}) = \mathbf{0}$ for more time periods than in the original interval $t = 1, \ldots, T$. Rather than doing it iteratively like FT, the solutions for all time periods $t = 1, \ldots, T, T+1, \ldots, T+k$ are found in one go by solving the stacked-time equation system

$$\begin{aligned}
\mathbf{f}(\mathbf{y}_1, \mathbf{y}_0, \mathbf{y}_2, \mathbf{x}_1, \hat{\beta}) &= \mathbf{0} \\
&\vdots \\
\mathbf{f}(\mathbf{y}_T, \mathbf{y}_{T-1}, \mathbf{y}_{T+1}, \mathbf{x}_T, \hat{\beta}) &= \mathbf{0} \\
\mathbf{f}(\mathbf{y}_{T+1}, \mathbf{y}_T, \mathbf{y}_{T+2}, \mathbf{x}_{T+1}, \hat{\beta}) &= \mathbf{0} \\
&\vdots \\
\mathbf{f}(\mathbf{y}_{T+k-1}, \mathbf{y}_{T+k-2}, \mathbf{y}_{T+k}, \mathbf{x}_{T+k}, \hat{\beta}) &= \mathbf{0} \\
\mathbf{f}(\mathbf{y}_{T+k}, \mathbf{y}_{T+k-1}, \bar{\mathbf{y}}_{T+k+1}, \mathbf{x}_{T+k}, \hat{\beta}) &= \mathbf{0}
\end{aligned} \tag{9}$$

given $\mathbf{y}_0$, $\mathbf{x}_{T+1}, \ldots, \mathbf{x}_{T+k}$, and $\bar{\mathbf{y}}_{T+k+1}$. By k being set sufficiently far into the future it is assumed that a reasonably chosen $\bar{\mathbf{y}}_{T+k+1}$ hardly affects the solution path in the original time interval, (Hall and Henry (1986) list four ways of deciding terminal values). All equations in the original model are duplicated for each simulation period, and each timing of a variable is considered to be a distinct variable itself. Rather than solving the original model period by period, the stacked (and then "static") model is solved for all periods simultaneously. Time is effectively removed from the solution of the model by integrating it into the stacked structure. The system of equation systems (9) illustrates that the numerical task of solving a small forward-looking model can be equivalent to solving a very large backward-looking model.

The benefit of time-stacking is elimination of the time dimension and the problem of future values. The cost is increased size of the model, which can be substantial. For a linear model the stacking leads to a huge matrix which is block tridiagonal, and can be effectively inverted with sparse matrix techniques, see e.g. Hollinger (1996). For a nonlinear model approximative solution techniques have to be applied. The stacking eliminates the Type II iterations in the FT-method, as the solutions for all periods are found directly by Type I iterations. In the TROLL software Stacked-Time solutions can optionally replace Type II iterations in the FT-macro. To check for the independence of the solution on the terminal conditions the Type III iterations have to be performed. In practice, iterations on k are often not performed. One rather chooses an assumingly large enough value of k right away, and then does not bother about the Type III iterations.

5 Stochastic simulation of a forward-looking model

In this paper the purpose of stochastic simulation is motivated by the need to update agents' expectations about model output in light of their recognized uncertainty about model input. In stochastic simulation of the backward-looking model (2) the stochastic nature of parameter estimates reflects the model builder's uncertain knowledge of their true values. The errors on the other hand are standard stochastic variables in the model. Despite the different interpretation of their stochasticity, the errors and the estimates were treated the same way in the simulations. When it comes to stochastic simulation of the forward-looking model things get more complicated. Therefore let us first look at simulation with stochastic error terms only.

We start with a single dynamic simulation with stochastic shocks (that is one replication n where the superscript $^{(n)}$ has been dropped to simplify notation):

$$\tilde{\mathbf{y}}_t = \mathbf{g}(\tilde{\mathbf{y}}_{t-1}, \bar{\mathbf{y}}_{t+1}, \mathbf{x}_t, \tilde{\mathbf{u}}_t, \hat{\beta}), \quad \tilde{\mathbf{y}}_0 = \mathbf{y}_0, \quad t = 1, \ldots, T+k \tag{10}$$

What separates this simulation from the deterministic simulation (8) is that the disturbances $\tilde{\mathbf{u}}_t$ are not set equal to their expectations $\mathbf{0}$. The disturbances perturb the solutions relative to the expectations, hence the simulated values $\tilde{\mathbf{y}}_{t+1}$ do not coincide with the expected values $\bar{\mathbf{y}}_{t+1}$. This is a complicating factor compared to the deterministic simulation based on certainty-equivalence. The realization and the expectation cannot be solved to-

gether as one and the same variable. Extra simulations are needed for the expectations. Once $\bar{\mathbf{y}}_{t+1}$ is quantified, $\tilde{\mathbf{y}}_t$ can be simulated with $\tilde{\mathbf{u}}_t$ input instead of $\mathbf{0}$.

The simulations of the expectation vectors $\bar{\mathbf{y}}_{t+1}, t = 1, \ldots, T+k-1$ (the terminal $\bar{\mathbf{y}}_{T+k+1}$ is not simulated, it is set exogenously) cannot be done in one go. The reason is that the agents form their expectations of future values based on the information available at the time the expectations are formed. When a new value $\mathbf{y}_t$ is realized, that value is different from what the agents expected because of the shocks $\mathbf{u}_t$. Consequently, the agents update their expectations based on this new information. This implies that for each viewpoint date at the beginning of period t it is necessary to perform simulations to update the expectations $\bar{\mathbf{y}}_{t+1}, \bar{\mathbf{y}}_{t+2}, \ldots$. Updating after each period has one advantage though. We only need the expectation values for period $t+1$. But to reduce the influence of terminal values the model needs to be solved for k periods ahead of $t+1$, that is for $s = t, \ldots, t+k+1$.

One remaining big issue concerns the way the agents form their expectations. Under the rational expectations hypothesis the agents know the model $\mathbf{f}$ and the parameters $\hat{\beta}$. The latest realizations $\tilde{\mathbf{y}}_{t-1}$ are known, and the exogenous values $\mathbf{x}_t$ can be considered known at the start of period t. When it comes to the temporal shocks $\mathbf{u}_t$, there are three possible interpretations. The next three subsections discuss these three possibilities in turn.

5.1 Agents have perfect foresight

The most demanding assumption on behalf of the economic agents implies the easiest solution method. If we assume that the agents know the exact value of the future shocks $\mathbf{u}_{t+1}$, then $\mathsf{E}[\mathbf{y}_{t+1} \mid \mathcal{I}_{t-1}] = \mathbf{y}_{t+1}$. In this case what separates each replicated stochastic solution path from the single deterministic solution path based on certainty-equivalence is that the (expected) zero shocks in (8) are replaced by a random shock $\mathbf{u}_t$. For a single dynamic simulation the stacked-time equation system (9) is replaced by

$$\begin{aligned} \mathbf{f}(\mathbf{y}_1, \mathbf{y}_0, \mathbf{y}_2, \mathbf{x}_1, \hat{\beta}) &= \mathbf{u}_1 \\ &\vdots \\ \mathbf{f}(\mathbf{y}_{T+k}, \mathbf{y}_{T+k-1}, \bar{\mathbf{y}}_{T+k+1}, \mathbf{x}_{T+k}, \hat{\beta}) &= \mathbf{u}_{T+k}. \end{aligned}$$

A full stochastic dynamic simulation experiment performs $n = 1, \ldots, N$ deterministic dynamic simulations. Each replicated dynamic simulation $(\mathbf{y}_1^{(n)}, \ldots, \mathbf{y}_{T+k}^{(n)})$ is driven by a unique sequence or path of shocks $(\mathbf{u}_1^{(n)}, \ldots, \mathbf{u}_{T+k}^{(n)})$. The extended exogenous path $\mathbf{x}_{T+1}, \ldots, \mathbf{x}_{T+k}$, the start vector $\mathbf{y}_0$ and the single terminal expectation vector $\bar{\mathbf{y}}_{T+k+1}$ remain fixed over the replications. In a stochastic simulation the shock paths will cause the solution paths to diverge, while the fixed terminal expectation vector will try to pull them together towards the end of the simulation. It is thus important to make the extension of k periods long enough to avoid significant influence of the terminal vector on the solutions in the original time interval $t = 1, \ldots, T$.

Algorithm 1 showed the recursive structure of solving a stochastic backward-looking model. The simultaneous model was solved recursively in the second For t-loop, as a multivariate function of variables predetermined at each period t. Algorithm 3 below differs from algorithm 1 by the lack of a For t-loop in which the model is solved recursively. The model has to be solved simultaneously for all variables *and* all periods.

Algorithm 3: Stochastic simulation of a perfect foresight model

For $n = 1, 2, \ldots, N$ replications of a single dynamic simulation

$\tilde{\mathbf{u}}_1^{(n)}, \ldots \tilde{\mathbf{u}}_{T+k}^{(n)} \sim \mathsf{I}_{n,t}\mathsf{IN}(\mathbf{0}, \widehat{\mathbf{\Sigma}})$,

Solve by FT or by ST:

$\tilde{\mathbf{y}}_1^{(n)} = \mathbf{g}(\mathbf{y}_0, \tilde{\mathbf{y}}_2^{(n)}, \mathbf{x}_1, \tilde{\mathbf{u}}_1^{(n)}, \hat{\beta}), .$

$\vdots$

$\tilde{\mathbf{y}}_{T+k}^{(n)} = \mathbf{g}(\tilde{\mathbf{y}}_{T+k-1}^{(n)}, \bar{\mathbf{y}}_{T+k+1}, \mathbf{x}_{T+k}, \tilde{\mathbf{u}}_{T+k}^{(n)}, \hat{\beta})$.

5.2 Agents know only the expectation of the shocks

The least demanding assumption on behalf of the economic agents is to assume that they only know the expected value of the shocks, $\mathsf{E}\mathbf{u}_t = \mathbf{0}$. Their expectations can then be approximated by separate deterministic dynamic simulations (8) based on certainty-equivalence. With viewpoint t, $\bar{\mathbf{y}}_{t+1}$ is found by simulating $t + k + 1$ periods:

$$\begin{aligned} \bar{\mathbf{y}}_t &= \mathbf{g}(\tilde{\mathbf{y}}_{t-1}, \bar{\mathbf{y}}_{t+1}, \mathbf{x}_t, \mathbf{0}, \hat{\beta}), \\ &\vdots \\ \bar{\mathbf{y}}_{t+k} &= \mathbf{g}(\bar{\mathbf{y}}_{t+k-1}, \bar{\mathbf{y}}_{t+k+1}, \mathbf{x}_{t+k}, \mathbf{0}, \hat{\beta}), \end{aligned}$$

or, in a more compact notation,

$$\bar{\mathbf{y}}_s = \mathbf{g}(\bar{\mathbf{y}}_{s-1}, \bar{\mathbf{y}}_{s+1}, \mathbf{x}_s, \mathbf{0}, \hat{\beta}), \quad \bar{\mathbf{y}}_{t-1} = \tilde{\mathbf{y}}_{t-1}, \quad \bar{\mathbf{y}}_{t+k+1} = \bar{\mathbf{y}}_{t+k+1}^1, \quad s = t, \ldots, t+k. \tag{11}$$

This single replication is a deterministic dynamic simulation like (8), only shorter. For each viewpoint the expectations can be simulated deterministically. Out of the $k+1$ simulated vectors only $\bar{\mathbf{y}}_{t+1}$ is used. Then with $\bar{\mathbf{y}}_{t+1}$ estimated by this separate deterministic dynamic simulation over k periods, the stochastic realization $\tilde{\mathbf{y}}_t$ can be simulated just like the backward-looking model.

The complete stochastic dynamic simulation of the forward-looking model consists of N replicated dynamic simulations over $T + k$ periods. Each dynamic simulation (10) is a possible outcome of the dynamic process of the (model) economy. During the unfolding of the process the agents update their expectation. This is reflected by the separate and shorter dynamic simulations (11) that we have to do for each viewpoint $t = 1, \ldots T + k$, in each replication (10). The replicated dynamic simulations are different because of different outcomes of the residual process and because the agents form different expectations based on different realizations of the model economy.

Each viewpoint-t simulation of the expectations path k periods into the future needs a terminal value $\bar{\mathbf{y}}_{t+k+1}^1$. Instead of guessing terminal vectors for each viewpoint-t simulation of each replication (n), one can perform a deterministic dynamic simulation with

certainty-equivalence (8) through all periods $t = 1, \ldots, T + k$, to provide a set of terminal values (the final terminal value $\bar{\mathbf{y}}^1_{T+k+1}$ still needs to be set exogenously). These values are simulated once from viewpoint $t = 1$, and might appear biased" from later viewpoints of simulated realizations. But that causes no problem. We are only using the vector that is $k+1$ periods ahead of the current period t as a terminal value for the simulation of the expectation of the next period $t + 1$. It does not matter if the values $\bar{\mathbf{y}}^1_{t+1}, \ldots, \bar{\mathbf{y}}^1_{t+k}$ in between are off. They have served their purpose as previous terminal values. The horizon k should be set large enough for the period-t solution not to be too sensitive to biased" terminal values $\bar{\mathbf{y}}^1_{t+k+1}$.

Below we sketch an algorithm for the case where the agents know the expected value of the shocks and form their expectations as if they were doing certainty-equivalence simulations. The initial deterministic simulation of the stacked-time model (9) requires the solution for $T + k$ time specific vectors. A single replication (n) of the stochastic simulation requires in addition T times the solution of k time specific vectors plus a final simulation to get the wanted solution. This amounts to simulating $NT(k + 1)$ vectors in addition to the $T + k$ vectors of an initial deterministic simulation. This is more clearly seen from the algorithm below.

Algorithm 4: Stochastic simulation with zero expectations

Choose a sufficient extension length k and a terminal expectations vector $\bar{\mathbf{y}}^1_{T+k+1}$,

Extend the exogenous path with $\mathbf{x}_{T+1}, \ldots, \mathbf{x}_{T+k}$,

Solve the stacked-time model

$$\begin{aligned} \mathbf{f}(\bar{\mathbf{y}}^1_1, \mathbf{y}_0, \bar{\mathbf{y}}^1_2, \mathbf{x}_1, \hat{\beta}) &= \mathbf{0} \\ &\vdots \\ \mathbf{f}(\bar{\mathbf{y}}^1_{T+k}, \bar{\mathbf{y}}^1_{T+k-1}, \bar{\mathbf{y}}^1_{T+k+1}, \mathbf{x}_{T+k}, \hat{\beta}) &= \mathbf{0} \end{aligned}$$

to approximate the model consistent or rational expectation path $\bar{\mathbf{y}}^1_{1+k+1}, \ldots, \bar{\mathbf{y}}^1_{T+k}$ to be used as terminal expectations $\bar{\mathbf{y}}^1_{t+k+1}$ in the last line of of the stacked-time model in the For t-loop below,

For $n = 1, \ldots, N$ replications of a single dynamic simulation

Let $\tilde{\mathbf{y}}^{(n)}_0 = \mathbf{y}_0$,

For $t = 1, \ldots, T$ periods within a single dynamic replication (n)

Solve the stacked-time model

$$\begin{aligned} \mathbf{f}(\bar{\mathbf{y}}^{(n)}_t, \tilde{\mathbf{y}}^{(n)}_{t-1}, \bar{\mathbf{y}}^{(n)}_{t+1}, \mathbf{x}_t, \hat{\beta}) &= \mathbf{0} \\ &\vdots \\ \mathbf{f}(\bar{\mathbf{y}}^{(n)}_{t+k}, \bar{\mathbf{y}}^{(n)}_{t+k-1}, \bar{\mathbf{y}}^1_{t+k+1}, \mathbf{x}_{t+k}, \hat{\beta}) &= \mathbf{0} \end{aligned}$$

to approximate agents' model consistent expectation path $\bar{\mathbf{y}}^{(n)}_t, \ldots, \bar{\mathbf{y}}^{(n)}_{t+k}$ given the previous simulation $\tilde{\mathbf{y}}^{(n)}_{t-1}$ and the terminal expectation $\bar{\mathbf{y}}^1_{t+k+1}$,

keep only $\bar{\mathbf{y}}^{(n)}_{t+1}$,

draw one random vector $\tilde{\mathbf{u}}_t^{(n)} \sim \mathsf{N}(\mathbf{0}, \widehat{\mathbf{\Sigma}})$,
given $\bar{\mathbf{y}}_{t+1}^{(n)}$ and $\tilde{\mathbf{u}}_t^{(n)}$ simulate the period-t specific solution

$$\tilde{\mathbf{y}}_t^{(n)} = \mathbf{g}(\tilde{\mathbf{y}}_{t-1}^{(n)}, \bar{\mathbf{y}}_{t+1}^{(n)}, \mathbf{x}_t, \tilde{\mathbf{u}}_t^{(n)}, \hat{\beta}). \tag{12}$$

End For t-loop (returns a solution path $(\tilde{\mathbf{y}}_t^{(n)})_{t=1}^T$ for replication n),
End For n-loop (returns a bundle of N solution paths $\{(\tilde{\mathbf{y}}_t^{(n)})_{t=1}^T\}_{n=1}^N$).

Algorithm 4 solves the model $NT(k+1) + (T+k)$ times. This is a lot more than the $N(T+k)$ solutions of Algorithm 3. The $N(T-1)k + (T+k)$ extra solutions are due to agents updating of expectations when they observe the realized values of each period.

5.3 Agents know the distribution of the shocks

An alternative to agents' limited knowledge of the shocks is to assume that (i) agents know not only the expectation but rather the whole distribution of the shocks, $\mathbf{u}_t \sim \mathsf{IID}(\mathbf{0}, \mathbf{\Sigma})$, and in addition that (ii) at each viewpoint t they form their future expectations *as if* they solve

$$\begin{aligned}
\mathsf{E}_t\mathbf{y}_t &= \mathsf{E}_t\mathbf{g}(\mathbf{y}_{t-1}, \mathsf{E}_t\mathbf{y}_{t+1}, \mathbf{x}_t, \mathbf{u}_t, \beta), \\
&\vdots \\
\mathsf{E}_t\mathbf{y}_{t+k} &= \mathsf{E}_t\mathbf{g}(\mathsf{E}_t\mathbf{y}_{t+k-1}, \mathsf{E}_t\mathbf{y}_{t+k+1}, \mathbf{x}_{t+k}, \mathbf{u}_{t+k}, \beta), \\
&\vdots
\end{aligned} \tag{13}$$

To approximate this expectation formation at viewpoint t, starting from simulation $\tilde{\mathbf{y}}_{t-1}$, we would have to simulate

$$\begin{aligned}
\bar{\mathbf{y}}_t &= \mathsf{E}_t\mathbf{g}(\tilde{\mathbf{y}}_{t-1}, \bar{\mathbf{y}}_{t+1}, \mathbf{x}_t, \tilde{\mathbf{u}}_t, \hat{\beta}), \\
&\vdots \\
\bar{\mathbf{y}}_{t+k} &= \mathsf{E}_t\mathbf{g}(\bar{\mathbf{y}}_{t+k-1}, \bar{\mathbf{y}}_{t+k+1}, \mathbf{x}_{t+k}, \tilde{\mathbf{u}}_{t+k}, \hat{\beta}),
\end{aligned}$$

where the expectation is with respect to the simulation residual $\tilde{\mathbf{u}}_t \sim \mathsf{I}_{n,t}\mathsf{IN}(\mathbf{0}, \widehat{\mathbf{\Sigma}})$. A more compact notation, which corresponds to (11) is

$$\bar{\mathbf{y}}_s = \mathsf{E}_t\mathbf{g}(\bar{\mathbf{y}}_{s-1}, \bar{\mathbf{y}}_{s+1}, \mathbf{x}_s, \tilde{\mathbf{u}}_s, \hat{\beta}), \quad \bar{\mathbf{y}}_{t-1} = \tilde{\mathbf{y}}_{t-1}, \quad \bar{\mathbf{y}}_{t+k+1} = \bar{\mathbf{y}}_{t+k+1}^1, \quad s = t, \ldots, t+k.$$

The obvious problem is that in order to simulate $\bar{\mathbf{y}}_s$ we need to know $\bar{\mathbf{y}}_{s+1}$. Let us assume that we have preliminary estimates of every expectation $\bar{\mathbf{y}}_s$, for instance from a deterministic certainty-equivalence simulation. Then we could simulate

$$\tilde{\mathbf{y}}_t^{+(m)} = \mathbf{g}(\tilde{\mathbf{y}}_{t-1}, \bar{\mathbf{y}}_{t+1}, \mathbf{x}_t, \tilde{\mathbf{u}}_t^{(m)}, \hat{\beta}) \quad \text{for } m = 1, \ldots, M, \tag{14}$$

and

$$\tilde{\mathbf{y}}_t^{-(m)} = \mathbf{g}(\tilde{\mathbf{y}}_{t-1}, \bar{\mathbf{y}}_{t+1}, \mathbf{x}_t, -\tilde{\mathbf{u}}_t^{(m)}, \hat{\beta}) \quad \text{for } m = 1, \ldots, M, \tag{15}$$

with antithetic residuals $\pm\tilde{\mathbf{u}}_t^{(m)}$, and calculate the sample mean

$$\bar{\mathbf{y}}_t = \frac{\overline{\tilde{\mathbf{y}}_t^{+(m)}} + \overline{\tilde{\mathbf{y}}_t^{-(m)}}}{2} = \frac{1}{2M}\sum_{m=1}^{2M}\left(\tilde{\mathbf{y}}_t^{+(m)} + \tilde{\mathbf{y}}_t^{-(m)}\right). \tag{16}$$

Using antithetic residuals is a standard variance reducing technique for estimating expectations which in this context allows a smaller number of replications M to achieve a given precision in the simulated expectations, see for instance Ripley (1987) and Calzolari (1979). We could continue with simulating (14)–(15) and calculating (16) for viewpoints $t, \ldots, t+k$. Replacing the preliminary estimates with the new estimates of expectations, the procedure could be iterated to improve the single vector $\bar{\mathbf{y}}_{t+1}$ needed at each viewpoint t in the replication (10). Allowing the agents to know the distribution of the error term and to form expectations *as if* they solve (13) imply that we have to perform stochastic simulations to estimate expectations within the overall stochastic simulation to estimate the distribution of the endogenous variables. This is clearly a very demanding task which we shall not consider further.

5.4 Uncertain knowledge of β

We now look at how to incorporate into the simulations uncertainty in our knowledge of the true values of the model parameters β. So far we have proceeded as if both the modeller and the economic agents knew the true values $\beta = \hat{\beta}$. Under the rational expectations hypothesis the agents indeed do know the true values. But the modeller only knows the best estimates and their sample distribution. To simulate the model under this perspective, we may assume that the estimates $\hat{\beta}$ are the true values, and that the sample distribution $\mathsf{N}(\hat{\beta}, \widehat{\mathbf{\Omega}})$ reflects the imprecise knowledge of the modeller. Then we only need to make two changes to algorithm 4. First we have to add

> Draw a random vector $\tilde{\beta}^{(n)} \sim \mathsf{IN}(\hat{\beta}, \widehat{\mathbf{\Omega}})$,

to the beginning of the For $t = 1, \ldots, T$ loop. Then we have to modify the final period-specific simulation (12) to use this parameter vector:

$$\tilde{\mathbf{y}}_t^{(n)} = \mathbf{g}(\tilde{\mathbf{y}}_{t-1}^{(n)}, \bar{\mathbf{y}}_{t+1}^{(n)}, \mathbf{x}_t, \tilde{\mathbf{u}}_t^{(n)}, \tilde{\beta}^{(n)}).$$

The two stacked-time simulations in algorithm 4 that mimic the expectation formations continue to use the "true" parameter vector $\hat{\beta}$. The agents form their expectations knowing the true value of the parameter vector. There is no uncertainty in their expectations. The uncertainty in the model simulations come from the error terms and the modeller's uncertain knowledge of β in (12).

The best sample-based estimate of the true parameter vector is $\hat{\beta}$. As such this estimate is used in the simulations of the expectations. But uncertainty in the modellers knowledge of the true parameter implies the same uncertainty about the (true) value the agents use. It is thus possible to simulate the parameter vector for the agents too. Then there are two possible ways to proceed. We may draw a new parameter vector

for each replication (n) and use it in simulating both the expectation path $\bar{\mathbf{Y}}^{(n)}$ and the solution path $\tilde{\mathbf{Y}}^{(n)}$, just like we did with $\hat{\beta}$ in the residual simulations in subsections 5.1, 5.2 and 5.3. Or, recognizing that we do not know the true value the agents use, we may simulate their parameter vector in addition to the modeller's vector of estimates. That amounts to replicating the procedure in the paragraph above: Draw parameter vectors of the agents $\hat{\beta}_A^{(n)} \sim \mathsf{I}_n\mathsf{IN}(\hat{\beta}, \widehat{\mathbf{\Omega}})$, and for each value $\hat{\beta}_A^{(n)}$ simulate the expectations. For each replication n do stochastic simulation with $m = 1, \ldots, M$ replications of the modellers vector of estimates independently drawn from the sample distribution of the estimate $\tilde{\beta}_M^{(n)} \sim \mathsf{I}_n\mathsf{IN}(\hat{\beta}, \widehat{\mathbf{\Omega}})$. The subscript $_A$ and $_M$ denote the values of the agents and the modeller. The M replications of model solutions are nested within the N replications of agents expectations. For large N this procedure can be approximated by independent draws of the two vectors within the same single replication, saving the expensive nested simulation. We summarize the three alternatives:

(a) Let $\hat{\beta}_A = \hat{\beta}$ and draw $\tilde{\beta}_M^{(n)} \sim \mathsf{I}_n\mathsf{IN}(\hat{\beta}, \widehat{\mathbf{\Omega}})$,

(b) Draw $\hat{\beta}_A^{(n)} = \tilde{\beta}_M^{(n)} = \tilde{\beta}^{(n)} \sim \mathsf{I}_n\mathsf{IN}(\hat{\beta}, \widehat{\mathbf{\Omega}})$,

(c) Draw independently of each other $\hat{\beta}_A^{(n)} \sim \mathsf{I}_n\mathsf{IN}(\hat{\beta}, \widehat{\mathbf{\Omega}})$ and $\tilde{\beta}_M^{(n)} \sim \mathsf{I}_n\mathsf{IN}(\hat{\beta}, \widehat{\mathbf{\Omega}})$.

The three alternatives are shown in an algorithm that is very similar to algorithm 4:

Algorithm 5: Stochastic simulation with random residuals and estimates

Choose a "sufficient" extension length k and a terminal expectations vector $\bar{\mathbf{y}}_{T+k+1}^1$,

Extend the exogenous path with $\mathbf{x}_{T+1}, \ldots, \mathbf{x}_{T+k}$,

For $n = 1, \ldots, N$ replications of a single dynamic simulation

Let $\tilde{\mathbf{y}}_0^{(n)} = \mathbf{y}_0$ and $\bar{\mathbf{y}}_{T+k+1}^{(n),1} = \bar{\mathbf{y}}_{T+k+1}^1$,

Either

(a) let $\hat{\beta}_A = \hat{\beta}$ and draw $\tilde{\beta}_M^{(n)} \sim \mathsf{I}_n\mathsf{IN}(\hat{\beta}, \widehat{\mathbf{\Omega}})$,

(b) draw one random vector $\hat{\beta}_A^{(n)} \equiv \tilde{\beta}_M^{(n)} \sim \mathsf{I}_n\mathsf{IN}(\hat{\beta}, \widehat{\mathbf{\Omega}})$ or

(c) draw two independent random vectors $\hat{\beta}_A^{(n)}, \tilde{\beta}_M^{(n)} \sim \mathsf{I}_n\mathsf{IN}(\hat{\beta}, \widehat{\mathbf{\Omega}})$,

Solve the stacked-time model

$$\begin{aligned} \mathbf{f}(\bar{\mathbf{y}}_1^{(n),1}, \mathbf{y}_0, \bar{\mathbf{y}}_2^{(n),1}, \mathbf{x}_1, \tilde{\beta}_A^{(n)}) &= \mathbf{0} \\ &\vdots \\ \mathbf{f}(\bar{\mathbf{y}}_{T+k}^{(n),1}, \bar{\mathbf{y}}_{T+k-1}^{(n),1}, \bar{\mathbf{y}}_{T+k+1}^1, \mathbf{x}_{T+k}, \tilde{\beta}_A^{(n)}) &= \mathbf{0} \end{aligned}$$

to get an initial rational expectation path $\bar{\mathbf{y}}_{1+k+1}^{(n),1}, \ldots, \bar{\mathbf{y}}_{T+k}^{(n),1}$ to be used as terminal expectations $\bar{\mathbf{y}}_{t+k+1}^{(n),1}$ in the last line of the stacked-time model in the For t-loop below,

For $t = 1, \ldots, T$ periods within a single dynamic simulation

Solve the stacked-time model

$$\begin{aligned}\mathbf{f}(\bar{\mathbf{y}}_t^{(n)}, \tilde{\mathbf{y}}_{t-1}^{(n)}, \bar{\mathbf{y}}_{t+1}^{(n)}, \mathbf{x}_t, \tilde{\beta}_A^{(n)}) &= \mathbf{0} \\ &\vdots \\ \mathbf{f}(\bar{\mathbf{y}}_{t+k}^{(n)}, \bar{\mathbf{y}}_{t+k-1}^{(n)}, \bar{\mathbf{y}}_{t+k+1}^{(n),1}, \mathbf{x}_{t+k}, \tilde{\beta}_A^{(n)}) &= \mathbf{0}\end{aligned}$$

to approximate the model consistent or rational expectation path $\bar{\mathbf{y}}_t^{(n)}, \ldots, \bar{\mathbf{y}}_{t+k}^{(n)}$ given the previous simulation $\tilde{\mathbf{y}}_{t-1}^{(n)}$ and the terminal expectation $\bar{\mathbf{y}}_{t+k+1}$ from the initial stacked-time simulation,
keep $\bar{\mathbf{y}}_{t+1}^{(n)}$,
draw a random vector $\tilde{\mathbf{u}}_t^{(n)} \sim \mathsf{I}_{n,t}\mathsf{IN}(\mathbf{0}, \widehat{\boldsymbol{\Sigma}})$,
simulate the period-specific solution

$$\tilde{\mathbf{y}}_t^{(n)} = \mathbf{g}(\tilde{\mathbf{y}}_{t-1}^{(n)}, \bar{\mathbf{y}}_{t+1}^{(n)}, \mathbf{x}_t, \tilde{\mathbf{u}}_t^{(n)}, \tilde{\beta}_M^{(n)}).$$

End For t-loop (returns one solution path $(\tilde{\mathbf{y}}_t^{(n)})_{t=1}^T$ for replication n),
End For n-loop (returns a bundle of N solution paths $\{(\tilde{\mathbf{y}}_t^{(n)})_{t=1}^T\}_{n=1}^N$).

Algorithm 5 seems likely to find more dispersed solutions paths than algorithm 4 since the parameter vector of the agents also varies with the replications. The stochastic simulation produces T period-specific multivariate samples $\{\tilde{\mathbf{y}}_t^{(n)}\}_{n=1}^N$, one for each $t = 1, \ldots, T$. In the literature the main motivation for stochastic simulation appears to have been to assess bias in the deterministic simulation $\tilde{\mathbf{y}}_t$ (3) relative to a mean stochastic simulation,

$$\hat{\mathbf{b}}_t = bias(\tilde{\mathbf{y}}_t) = \frac{1}{N}\sum_{n=1}^{N} \tilde{\mathbf{y}}_t^{(n)} - \tilde{\mathbf{y}}_t, \tag{17}$$

and not so much to approximate the uncertainty in a deterministic solution by, say the standard deviation of the stochastic simulation,

$$\hat{\sigma}_t = st.dev(\tilde{\mathbf{y}}_t) = \sqrt{\frac{1}{N-1}\sum_{n=1}^{N}\left(\tilde{\mathbf{y}}_t^{(n)} - \bar{\mathbf{y}}_t\right)^2} \approx \sqrt{\left(\tilde{\mathbf{y}}_t^{(n)}\right)^2 - (\bar{\mathbf{y}}_t)^2}, \tag{18}$$

Deterministic bias appears to be small in operative macroeconometric models which are only mildly nonlinear and dominantly backward-looking, see for instance Fair (1984), Fisher and Salmon (1986) or Hall and Henry (1988). This need not be so for forward-looking models. In any case, uncertainty in the form of standard deviations or prediction intervals should be interesting too, and useful.

6 Concluding remarks

When a model contains just a few expectation variables, is mildly nonlinear and has neglectible shock persistence, one might be tempted to try a major shortcut to the above algorithms: Simulate the model deterministically once to get the expectations, then do

a regular stochastic simulation with "exogenous" expectations. Such a procedure might be attempted as a first approach when the simulation horizon is short. The obvious advantage is that programs for stochastic simulation of backward-looking models can be used. A potentially serious disadvantage is that the simulated samples seem likely to be both biased and underdispersed.

Deterministic and stochastic simulations of both backward- and forward-looking models are all conditional on the path of the vector of exogenous variables. In a forward-looking model this introduces a type of inconsistency. Unless they have perfect foresight the agents do not use actual values, but rather some expectations of the exogenous variables when solving for model consistent expectations of the endogenous variables. In ex ante simulations this inconsistency can be relieved (in an ad hoc way though) by adding white noise to smooth exogenous projections in the final solving for the endogenous variables. Solving for agents' expectations remains based on the smooth projections. For in-sample simulation the corresponding procedure can be inverted. Now the exogenous expectations can be approximated by smoothing the observed values. The final solving for the endogenous variables uses the realized values of the exogenous variables.
Ex ante simulations might involve stochastic simulation of perturbations to fixed smooth expectations paths for the exogenous variables in addition to stochastic simulation of the shocks and the parameter estimates. The in-sample simulations on the other hand entail only the use of two exogenous paths, the actual and the smoothed data. Despite the procedure being ad hoc, resulting increases in estimated standard errors of the solutions, or equivalently, widening of prediction intervals, might be significant and informative.

References

Armstrong, J., R. Black, D. Laxton and D. Rose (1998). A Robust Method for Simulating Forward-Looking Models. *Journal of Economic Dynamics and Control*, *122*, 489–501.

Calzolari, G. (1979). Antithetic Variates to Estimate the Simulation Bias in Non-Linear Models. *Economics Letters*, *4*, 323–328.

Davidson, R. and J. G. MacKinnon (1993). *Estimation and Inference in Econometrics*. Oxford University Press, New York, Oxford.

Fair, R. C. (1984). *Specification, Estimation and Analysis of Macroeconometric Models*. Harvard University Press, Cambridge, Massachusettes · London, England.

Fair, R. C. and J. B. Taylor (1983). Solutions and Maximum Likelihood Estimation of Dynamic Nonlinear Rational Expectations Models. *Econometrica*, *51*(4), 1169–1185.

Fair, R. C. and J. B. Taylor (1990). Full Information Estimation and Stochastic Simulation of Models with Rational Expectations. *Journal of Applied Econometrics*, *5*, 381–392.

Fisher, P. (1992). *Rational Expectations in Macroeconomic Models*. Kluwer Academic Publishers, Dordrecht.

Fisher, P., S. Holly and A. J. Hughes-Hallett (1986). Efficient Solution Techniques for Dynamic Non-Linear Rational Expectations Models. *Journal of Economic Dynamics and Control*, *10*, 1395–145.

Fisher, P. and M. Salmon (1986). On Evaluating the Importance of Non-linearity in Large Macroeconometric Models. *International Economic Review*, *27*, 625–646.

Gallo, G. M. and F. J. H. Don (1991). Forecast Uncertainty Due to Unreliable Data. *Economic and Financial Computing*, *1*, 49–69.

Gourieroux, C. and A. Monfort (1997). *Time Series and Dynamic Models*. Cambridge University Press, Cambridge.

Hall, S. G. and S. G. B. Henry (1986). A Dynamic Econometric Model of the UK with Rational Expectations. *Journal of Economic Dynamics and Control*, *10*, 219–223.

Hall, S. G. and S. G. B. Henry (1988). *Macroeconomic Modelling*. North Holland Publishing Company, Amsterdam.

Hendry, D. F. (1995). *Dynamic Econometrics*. Oxford University Press, Oxford.

Hollinger, P. (1990). *TROLL Program: Stochastic Simulator*. Intex Solutions Inc., Massachusetts.

Hollinger, P. (1996). The Stacked-Time Simulator in TROLL: A Robust Algorithm for Solving Forward-looking Models. Presented at the Second International Conference on Computing in Economics and Finance, Geneva, June 20-23, 1996.

Holly, S. and A. Hughes-Hallett (1989). *Optimal Control, Expectations and Uncertainty*. Cambridge University Press, Cambridge, Great Britain.

Juillard, M., D. Laxton, P. McAdam and H. Pioro (1998). An Algorithm Competition: First-order Iterations versus Newton-based Techniques. *Journal of Economic Dynamics and Control*, *22*, 1291–1318.

Ripley, B. D. (1987). *Stochastic Simulation*. John Wiley and Sons, New York.

Wallis, K. F. (1980). Econometric Implications of the Rational Expectations Hypothesis. *Econometrica*, *48*(1), 49–73.

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