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AKSEL BENUM
VETLE Ø. OPHEIM
ERIK S. WASBERG

# Estimations of the term premium on Norwegian government bonds

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# Estimations of the term premium on Norwegian government bonds

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*The term premium on government bonds has been estimated to be low in recent years. As a result, the additional cost of long-term borrowing has been relatively low. The term premium has edged up again over the past year. This* Memo *documents and discusses the model in Adrian, Crump and Moench (2013) that we use to assess developments in the term premium on Norwegian government bond yields*[1].

Key words: *Term premiums, government bonds, yield term structure*

## 1. Introduction

Norges Bank is responsible for managing Norway's government debt under a mandate from the Ministry of Finance. The primary objective of the Bank's government debt management is to meet the government's borrowing requirement at the lowest possible cost within given risk limits. At the same time, Norges Bank shall maintain a yield curve for maturities of up to ten years. Bonds with maturities of between one and twenty years cover a large part of the borrowing requirement.[2] However, it is not given that this strategy can be expected to result in the lowest possible interest expense for the government.

The expected additional cost of fixing interest costs long-term rather than short-term and/or at a floating rate is referred to as the term premium. A positive term premium makes the cost of long-term borrowing more expensive compared with short-term borrowing, all else equal.

Norges Bank Government Debt Management is able to manage the average time to refixing of the government's debt portfolio by using interest rate swaps[3]. If in Norges Bank's assessment, short-term borrowing results in lower borrowing costs than long-term borrowing, the Bank can reduce the average time to refixing of the debt portfolio by entering into interest rate swaps where the government receives a fixed rate and pays a floating rate. This means that the issuance strategy and the average time to refixing can be partially delinked.

Assessments related to the use of interest rate swaps are an important reason why Norges Bank Government Debt Management estimates the term premiums on Norwegian government bonds. The estimated level of and developments in the term

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[1] The views and conclusions expressed in this publication are the authors' own and do not necessarily reflect those of Norges Bank. Any errors or omissions are solely the responsibility of the authors. We thank Farooq Akram, Gaute Langeland, Kathrine Lund, Anders Svor, Olav Syrstad, Per Marius Pettersen and colleagues in Government Debt Management for useful input and comments.

[2] In recent years, one of the target indicators for government debt management has been that at least 50% of borrowing requirements must be met by the issuance of bonds with maturities of seven years or more. All Norwegian government bonds are fixed-rate instruments and pay no principal until maturity. The average time to refixing of the debt portfolio was 4.2 years at year-end 2022. Government Debt Management also buys back government bonds with less than one year to maturity, which increases effective government debt maturity.

[3] An interest rate swap is an agreement between two counterparties, where one of the counterparties pays a fixed rate and the other pays a floating rate.

premium on Norwegian government bonds are also useful for, among others, investors, financial system participants and the general public.

There are limited studies on developments in the term premium in Norwegian government bond yields, but it is our understanding that previous surveys include Wright (2011), Ceballos and Romero (2016), and de Lange, Risstad and Westgaard (2022). The latter uses, among other things, the model in Adrian, Crump and Moench (2013), which this *Memo* presents. Methods to estimate the term premium also includes Joslin et al (2014), Bauer et al (2015) and Bauer and Rudebusch (2016). Moe and Michelsen (2021) investigate different economic variables that can predict the Norwegian term premium in the period between 2009 and 2019. De Lange, Risstad and Westgaard (2023) examine term premiums in Norwegian swap rates.

This *Memo* documents the term premium model used by Norges Bank Government Debt Management, and the remainder of this *Memo* is structured as follows: Section 2 discusses the term premium as a concept, Section 3 outlines the theoretical and empirical method used to estimate the term premium, as well as some comments related to the choice of time period and a robustness check. Section 4 describes developments in the term premium since the turn of the millennium. The Appendix also contains a discussion of a cross-check of the results of our estimates.

# 2. The term premium

The expectations hypothesis[4] states that the yield on an investment with a long maturity will be equal to expected developments in short-term yields over the same period. According to this hypothesis, a two-year yield will be equal to the average of two subsequent one-year yields, or subsequent three-month yields that cover the period of this two-year yield. In other words, risk-free bonds with different maturities are perfect substitutes.

The expectations hypothesis may be a good starting point for understanding developments in long-term yields. However, the hypothesis disregards the fact that financial market participants may relate to different forms of risk, which will normally require a premium. For example, it can be assumed that participants will require a liquidity premium to invest in illiquid securities. It can also be assumed that a premium exists that increases with the bond issuer’s default risk, generally referred to as the credit risk premium. For bonds that are not inflation-indexed or inflation-protected, ie bonds that yield a return that compensates for inflation, investors will also require compensation for uncertainty related to future inflation.[5]

Normally, it would be reasonable to assume that investors will require a premium on investments with a long maturity. The price of a fixed-rate bond with a long residual maturity is more sensitive to interest rate movements than the price of a bond with a short residual maturity.[6] Correspondingly, a bond issuer will have to accept a higher yield in order to “lock in” the yield level compared with issuing a floating-rate bond or, if applicable, by rolling over several fixed-rate bonds with shorter maturities.

[4] See Fischer (1930) and Lutz (1940).

[5] Bernhardsen (2011) provides a thorough and applicable introduction to such premiums.

[6] However, some market participants will require long maturities on the asset side to balance liabilities far into the future. Long-term bond purchases may then be regarded as risk mitigation.

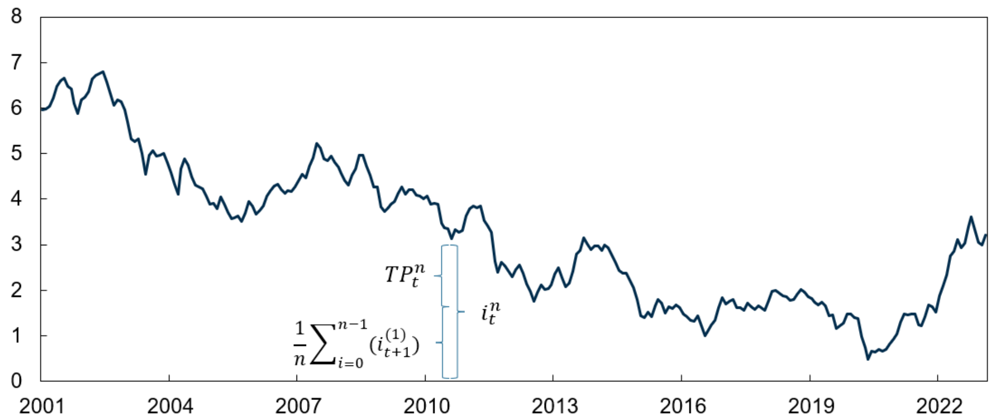
According to the expectations hypothesis, the compensation required by market participants beyond the yield is generally called the term premium.[7]

The relationship can be formulated as follows:

(2.1.) $$i_t^n = \frac{1}{n}\sum_{i=0}^{n-1} E_t\left(i_{t+i}^{(1)}\right) + TP_t^n$$

where $i_t^n$ is long-term yield with maturity of *n* years, $i_{t+i}^{(1)}$ is short-term yield at time *t+i*, and $\frac{1}{n}\sum_{i=0}^{n-1} E_t\left(i_{t+i}^{(1)}\right)$ is average expected short-term yield over the period *n, also referred as the risk neutral yield*. $TP_t^n$ is term premium on long-term yield with maturity *n*. Chart 1 shows this relationship in graphic form.

**Chart 1: A simplified presentation of the term premium**
Zero coupon yield and term premium. Percentage points




Source: Norges Bank

The term premium with maturity of *n* years can thus be defined as the difference between the yield with maturity of *n* years and average expected short-term yield over the next *n* years.

However, the expected short-term yield cannot be observed, and consequently, the term premium is also an unobservable variable. Nevertheless, there are different methods that seek to decompose bond yields into a risk-neutral component, which is an estimation of the average expected short-term yield, and a term premium.

# 3. Method

We use a model to decompose bond yields into a term premium and an expected short-term yield. Different methods and approaches are used to estimate the term premium. The choice of both model and horizon used in the estimation will have an impact on the estimated level of term premiums. The method primarily applied by Norges Bank Government Debt Management to estimate the term premium in Norwegian government bond yields follows Adrian, Crump and Moench (2013), hereinafter referred to as ACM.

[7] In principle, the definition of term premium, as used in this framework, could also contain other premiums, for example, a credit risk premium or a liquidity premium. However, in this article, we examine Norwegian government bond yields where the credit risk premium can be assumed to be equal to zero and the liquidity premium does not depend on the different maturities.

Kim and Wright (2005) is another recognised method. Other methods used to estimate the term premium, such as Vasicek (1977), Christensen and Rudebusch (2010), Hördahl and Tristani (2014) and Kopp and Williams (2018) are also worth mentioning. There are also models that recognise different macroeconomic variables to explain the level of bond yields where the term premium can be included, including Ang and Piazzesi (2003), Hördahl, Tristani and Vestin (2006), Rudebusch and Wu (2008) and Cochen, Hördahl and Xia (2018).

The ACM model is estimated using linear regressions, which makes the estimations technically easier to carry out. However, as explained in Cochen, Hördahl and Xia (2018), the model may "overreact" to changes in the yield level. This and other robustness checks are discussed in more detail in Section 5.

## 3.1 The ACM model

The term premium is the expected return on a long-term government bond in excess of the expected short-term yield over a horizon. The ACM model posits that the term structure of bond yields can be decomposed into a set of factors, principal components.[8] Furthermore, the ordinary least squares (OLS) method is used to estimate an average of expected short-term yields over the horizon, ie risk-neutral yields. The term premium is then calculated as the difference between the level of zero coupon yields and the average of future short-term yields.

A complete review of the model and the estimation method can be found in Adrian et al (2013) and is presented in the Appendix.

## 3.2 Further discussion of choice of horizon

The choice of the starting point of the estimation is highly relevant when using the ACM model. When new observations are added, the estimation of the term premiums is changed. The reason for this is that the weights used to estimate the principal components are estimated based on the full data set. When the data set is changed, the underlying data correlations will also change, which in turn may affect the weights.

Structural breaks in the data also affect the choice of horizon. An obvious example of a structural break that is relevant for the yields on Norwegian government securities is the introduction of a new monetary policy mandate in March 2001 with a 2.5 percent inflation target.[9] This is the main reason why Government Debt Management's estimations of the Norwegian term premium have 2001 as the start date.

Malik and Meldrum (2014) study the term structure of UK yields and set the starting point at May 1997, the same date as the Bank of England was given greater operational independence and a statutory inflation target**.** However, they show that the extension of the sample period back to when the UK first adopted an inflation targeting framework for monetary policy (October 1992) has little impact on the estimated premiums. The authors explain why one typically sees lower term premiums when sample periods are extended.

[8] Principal component analysis is a statistical method used to reduce a large number of variables to a smaller number of factors. The new factors are a linear transformation of the original variables and are constructed to minimise information loss from reducing the number of variables.
[9] However, it is claimed that in practice this inflation target was adopted from January 1999 (see Norges Bank (2022a)). In 2018, the inflation target was lowered from 2.5 percent to 2.0 percent.

The original analysis by Adrian et al (2013) uses data from January 1987. On its website, the Federal Reserve has chosen to publish term premiums based on a data set that goes back to 1961.[10]

The ACM model has inherent reversion effects or is mean-reverting for the short-term yield. When the current short-term yield is considerably lower than the average yield for the entire period, it is expected to revert to a higher level.[11] In the model, this long-term yield level is the same for the entire estimation period, which may be a weakness of the model.

Interest rates showed a falling trend in the period between the 1980s and 2010s. The reason for this may be complex.[12] If term premiums are estimated in a period of falling interest rates, the ACM model will indicate a risk-neutral yield that is too high, which results in a term premium that is lower than what otherwise would have been the case.

The opposite is true if the general level of interest rates rises and stabilises at a higher level. In such a scenario, the estimated term premium will be too high because the risk-neutral yield is assumed to be too low.

# 4. The term premium on Norwegian government bonds

Estimations using the ACM model suggest that the term premium on Norwegian government bonds fell considerably over several years. Between 2014 and 2021, the term premiums on Norwegian government bonds were largely estimated to be low or negative. Recently, term premiums have edged up again. This may be owing to the phasing out of quantitative easing in several key countries and uncertainty related to future developments in interest rates and inflation.

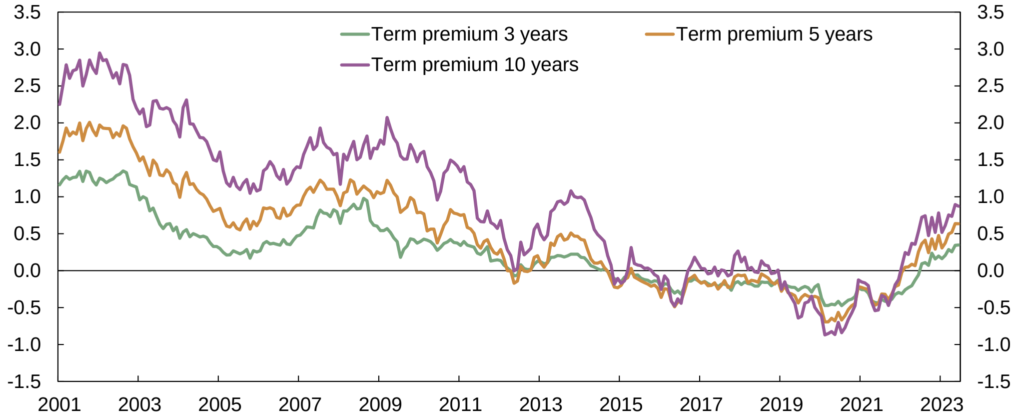
As shown in Chart 2, the estimated values of term premiums using the ACM method were close to zero or negative between 2014 and 2021. Moreover, the estimations show that term premiums were lower than on corresponding German government bond yields in the period to 2014 (Charts 2 and 4). Since 2022, term premiums on Norwegian government bonds have again been estimated to be positive.

[10] See Federal Reserve (2023)

[11] This assumes that the principal components in the yields' term structure can also be said to revert to a historical average. The Appendix to the article shows that the conditions for this apply.

[12] See Meyer, Ulvedal and Wasberg (2022) for a more detailed review.

**Chart 2: Term premium on Norwegian government bond yields with maturities of three, five and ten years**
Percentage points. March 2001 – August 2023



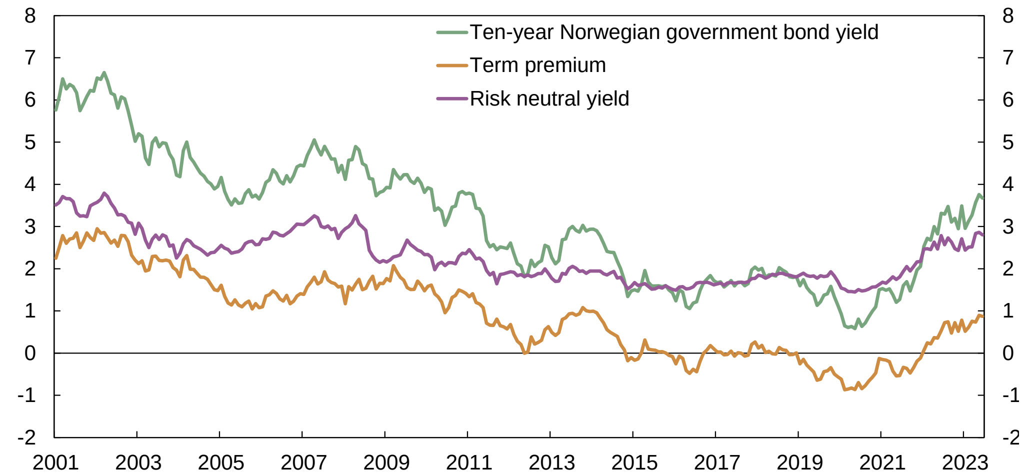

Source: Norges Bank

Chart 3 shows bond yield decomposed into a risk-neutral yield and term premium. The risk-neutral yield is the average of expected short-term yields over the given horizon. The risk-neutral yield in the ACM estimations has been falling somewhat over time but has risen since February 2021.

At the time of the publication of this *Memo*, the yield on the ten-year government bond is higher than the risk-neutral yield, which in this framework implies that the term premium is positive, albeit at a lower level than during the first decade after the turn of the millennium.

Between 2009 and 2019, the Norwegian ten-year government bond yield fell by around 200 basis points. The model indicates that close to 80% of the fall can be explained by a decline in the term premium.

**Chart 3: Estimated term premiums and neutral interest rates**
Percentage points. March 2001 – August 2023




Source: Norges Bank

For an extended period after the financial crisis, several of the leading central banks purchased government bonds and other securities to increase the degree of monetary accommodation. This measure is known as quantitative easing.[13] The choice to

[13] In cases where the policy rate is already at a low level, such quantitative easing may push down interest rates in the economy further to boost economic activity. Central banks' securities purchases, mainly government bonds, push up bond prices and push down yields. Owners of government bonds and other assets gain more wealth as a result of

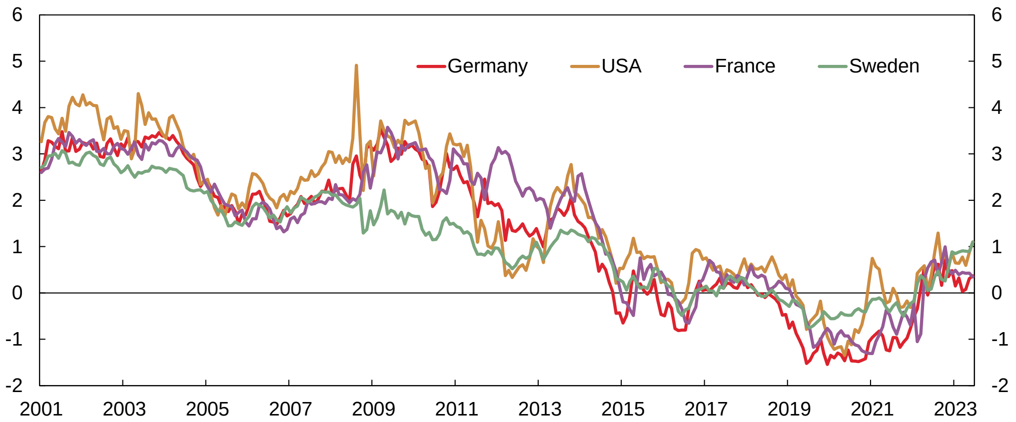
purchase securities may also be interpreted as a signal that yields will be kept low for a long period, which in isolation may reduce the term premium.

The literature on quantitative easing in other countries does not provide an unequivocal answer to the question of whether quantitative easing has had the strongest impact because it resulted in a fall in expected policy rates or through a lower term premium, as specified in Olsen (2019). Through portfolio effects and contagion from other fixed income markets, central bank policy has likely contributed to lower yields on Norwegian government bonds. Since 2022, a number of central banks have scaled back or ended their asset purchases.

Chart 4 shows developments in term premiums in ten-year government bond yields in selected countries. The term premium level varies to a certain extent, but developments over time appear to be fairly similar.

**Chart 4: Estimated term premiums in ten-year government bond yields in selected countries**

Percentage points. March 2001 – August 2023




Source: Norges Bank

# 5. Robustness checks of term premium estimations

Term premiums cannot be observed directly, and the estimations of the term premium will always be uncertain. Government Debt Management therefore performs a number of robustness checks to assess the estimations from the ACM model.

## 5.1 Bias-corrected term premium estimation

A challenge related to term premium estimations is that yields show a high degree of persistence, which is not sufficiently captured by the parameters when the estimation horizon is short, see Bauer et al (2012). This results in an estimation bias in the parameters and entails that the factors return too quickly to their historical averages.

increased values. The interest expenses of indebted households and firms will be reduced. When central banks purchase large volumes of government bonds, available market volumes are reduced. Investors will typically purchase other higher-risk assets or investment-grade government bonds from other countries, for example Norway.

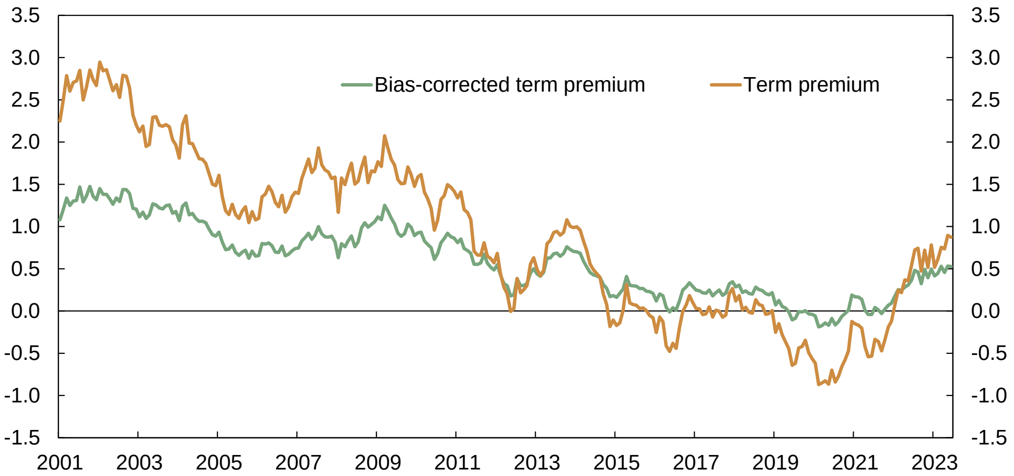
To correct for this bias, we use an approach described in Jennison (2017), whereby we re-estimate the vector autoregressive (VAR) model a number of times using randomly selected sub-periods.[14] On the basis of all the estimations, we calculate the average of the parameters $\bar{\hat{\Phi}}$. Finally, we calculate a bias-corrected matrix of the parameters.

(5.1.) $$\hat{\Phi}^{BC} = 2\hat{\Phi} - \bar{\hat{\Phi}}$$

The term premiums are then estimated with $\hat{\Phi}^{BC}$ as a starting point. Chart 5 shows the bias-corrected term premiums compared with the term premium estimated using the standard method. As in Jennison (2017) and Malik and Meldrum (2014), we see that the bias-correction reduces variation in the term premiums. As a result of the correction, the term premium estimates at the beginning of the estimation period are lower, while the term premium turns out to be higher in the period after 2014. This is because the term premium is estimated using an average of the re-estimated term premiums, resulting in smaller variations in the term premium.

Estimating the parameters on the basis of sub-periods results in a large number of parameter sets that can be used to estimate the term premium over the full horizon. The term premiums resulting from estimations based on sub-periods can then be compared with the estimation based on the full horizon.

**Chart 5: Bias-corrected term premiums**
Percentage points. 2001 – August 2023




Source: Norges Bank

## 5.2 Estimation uncertainty in the term premium estimations

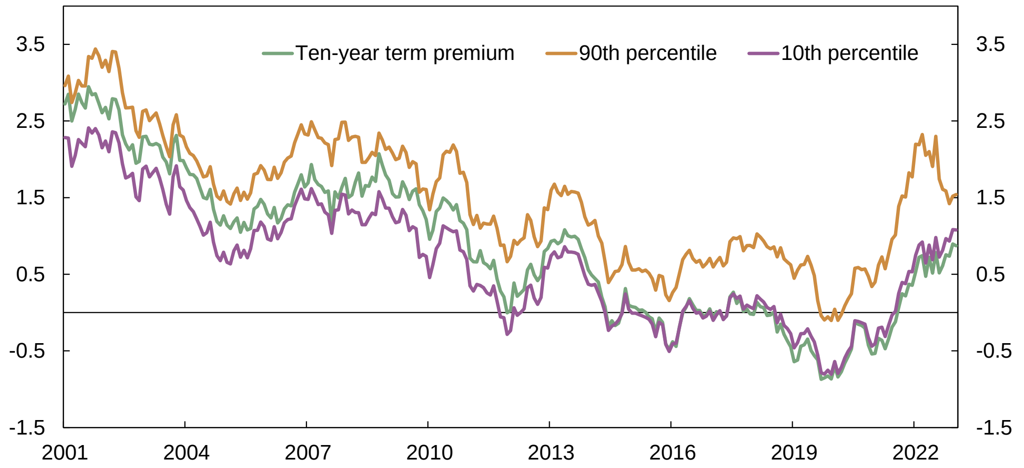
We also examine a method that seeks to estimate a sample space for term premium estimations made today compared with estimations made based on other underlying data. All the model parameters are re-estimated *n* times for a random sample of sub-periods of the full observation period.[15] Term premiums are then estimated for all the sub-periods. The results of the estimations are collected into a single dataset covering the entire period 2001-2023. The 10th and 90th percentiles of the estimations are shown in Chart 6 together with the estimations for the entire period.

Each estimation of the model based on a sub-period provides a unique set of coefficients. Using these coefficients to estimate the term premium for the entire period

[14] We have chosen to re-estimate 10 000 times.
[15] Here, *n* is set at 10 000.

provides us with a dataset with a large number of term premium forecasts for each point in time over the entire horizon. Ten percent of the estimations are below the 10th percentile graph at each point in n time, as shown in Chart 6. Correspondingly, 10 percent of the values are higher than the 90th percentile. In the chart, this is compared with the term premium estimation based on the entire horizon.

**Chart 6: Uncertainty intervals for the estimated term premiums**
Percentage points. August 2001 – August 2023




Source: Norges Bank

Estimations of the ten-year term premium are close to the 10th percentile for a long period, which indicates that the model generates a conservative estimation of the term premium when the entire horizon is used. The fact that the estimation of the term premium over the entire horizon is in below the lower part of the range likely reflects developments in yields. Most of the sub-periods covering latest observations in the overall dataset will show a lower yield level. Thus, the estimations for these sub-periods could result in a lower level of the estimated risk-neutral yield so that the term premium turns out to be higher.

# 6. Summary

Norges Bank Government Debt Management estimates the term premiums on Norwegian government bonds using a method developed by Adrian, Crump and Moench (2013). In isolation, positive term premiums represent a cost associated with long-term rather than short-term borrowing.

The model's estimation of the term premium on long-term government bonds fell significantly over several years and have remained low in recent years. This indicates that the additional cost associated with long-term borrowing has been relatively low. In the recent period, the term premiums on Norwegian government bonds have edged up again.

Term premium levels should be interpreted with caution. Decisions related to the model and dataset have a considerable impact on estimated term premium levels. However, the models can provide a reliable indication of developments in the term premiums.

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# Appendix

## ACM model

Here, the term premium is the premium on a long-term government bonds in excess of the expected short-term yield. The model aims to estimate the expected return, a convexity adjustment and a component that provides the term premium itself.

The ACM model assumes that the price of a bond is driven by a set of unobservable factors. These are made up of principal components,[16] which are assumed to change over time in accordance with process (3.1), a vector autoregressive (VAR) model, where shocks $v_{t+1}$ are normally distributed with an expected value equal to 0 and a variance-covariance matrix $\Sigma$.

(1) $$X_{t+1} = \mu + \phi X_t + v_{t+1}$$

The price of a *n* period bond at time *t*, ie $P_t^n$, is defined recursively as the expected product of a stochastic discounting factor at time $t+1$, ie $k_{t+1}$, and the price of the same bond after one period in $t+1$, as shown in (2).

(2) $$P_t^n = E_t[k_{t+1}P_{t+1}^{n-1}].$$

The stochastic discounting factor $k_t$, includes all relevant information on price setting and is used along the entire yield curve. The differences in yields are determined by investor perceptions of risk and expected changes in the stochastic discounting factor. Zero coupon yields are used to optimally compare yields.[17]

The stochastic discounting factor is a function of the risk-free interest rate $r_t$ and the market price of risk $\lambda_t$ at time $t$, as shown in (3).

(3) $$k_{t+1} = exp\left(-r_t - \frac{1}{2}\lambda'\lambda_t - \lambda'\Sigma^{-1/2}v_{t+1}\right).$$

The risk-free interest rate $r_t = \ln\ P_t^1$ is the interest rate with a tenor at the shortest unit of time. In the estimations performed by Norges Bank, this is one month.

The ACM model assumes that market pricing of risk $\lambda_t$ is linear depending on the unobservable factors, as shown in (4), where $\lambda_0$ is the time varying component of the market pricing of risk, and $\lambda_1$ is the component that varies over time.

(4) $$\lambda_t = \Sigma^{-1/2}(\lambda_0 + \lambda'_1 X_t),$$

The logarithm of the excess return from holding a bond maturing in $n$ periods from $t-1$ to $t$, beyond the risk-free interest rate, is given by (5).

(5) $$rx_{t+1}^{(n-1)} = \ln P_{t+1}^{(n-1)} - \ln P_t^n - r_t$$

[16] Principal component analysis is a statistical method used to reduce a high number of variables to a smaller number of factors. The new factors are a linear transformation of original variables and are constructed to minimise the loss of information from reducing the number of variables.

[17] Zero coupon yields are yields that are not influenced by the fact that bonds can have different coupons. Norges Bank estimates these yields by using a parametric method developed by Nelson and Siegel and developed further by Svensson, referred to as the NSS method. See Norges Bank (2023) for further details.

By combining the equations[18] and assuming that $rx_{t+1}^{(n-1)}$ and $v_{t+1}$ have a common normal distribution, the excess return from holding a bond can be decomposed into one element that is independent and one that is correlated with $v_{t+1}$ (see ACM 2013 for a complete derivation).

The excess return on a bond $rx_{t+1}^{(n-1)}$ can be further described as a function of the bond's expected return $\beta^{(n-1)\prime}(\lambda_0 + \lambda_1 X_t)$, a convexity adjustment $\frac{1}{2}\left(\beta^{(n-1)\prime}\Sigma\beta^{(n-1)} + \sigma^2\right)$, a component correlated with $v_{t+1}$, ie $\beta^{(n-1)\prime}v_{t+1}$, and an uncorrelated error term, as shown in (6).

$$(6)\qquad rx_{t+1}^{(n-1)} = \beta^{(n-1)\prime}(\lambda_0 + \lambda_1 X_t) - \tfrac{1}{2}\left(\beta^{(n-1)\prime}\Sigma\beta^{(n-1)} + \sigma^2\right) + \beta^{(n-1)\prime}v_{t+1} + e_{t+1}^{(n-1)}.$$

Equation (6) can then be re-written as (7), where $rx$ is a $N \times T$ matrix with excess return, $\beta$ is a $K \times N$ matrix with coefficients, $\iota_T$ and $\iota_N$ are $T \times 1$ vectors with 1, $X\_$ is a $K \times T$ matrix with time-lag values of the unobservable factors, $B^* = \left[vec\left(\beta^{(1)}\beta^{(1)\prime}\right) \dots vec(\beta^{(N)}\beta^{(N)\prime})\right]$ and has the variable $\times K^2$, $V$ is a $K \times T$ matrix with shocks, and $E$ a $N \times T$ matrix with a residual term.

$$(7)\qquad rx = \beta'(\lambda_0\iota_T' + \lambda_1 X\_) - \tfrac{1}{2}(B^* vec(\Sigma) + \sigma^2\iota_N)\iota_T' + \beta' V + E$$

## Estimating the ACM model

Government Debt Management estimates the model's parameters in three steps by using linear regressions as in Adrian et al (2013).

In the first step, we estimate equation (1) using the ordinary least square (OLS) method, where unobservable factors $X_t$ are decomposed into a predictable component and an estimate of the residual term $u_t$. The residual terms are inserted into matrix $\widehat{\mathrm{V}}$ and are used to estimate the unobservable factors' variance-covariance matrix $\hat{\Sigma}$.

In the second step, as shown in (8), we regress the excess return on a constant, the estimated residual terms $\widehat{\mathrm{V}}$, and lagged values of the unobservable factors.

$$(8)\qquad rx = a\iota_T' + \beta'\widehat{\mathrm{V}} + cX\_ + E.$$

The independent variables are organised into matrix $\tilde{Z}' = \left[\iota_T\,\widehat{\mathrm{V}}'\,X'\_\right]'$, so that the OLS estimator generates (9).

$$(9)\qquad \left[\hat{a}\,\hat{\beta}'\hat{c}\right] = rx\ \tilde{Z}'\left(\tilde{Z}\,\tilde{Z}'\right)^{-1}.$$

The residual terms from this regression are used to estimate the excess return variance-covariance matrix $\sigma^2$. From $\hat{\beta}$ we construct $\hat{B}^*$In the third step, risk parameters $\lambda_0$ and $\lambda_1$ are estimated. Adrian et al (2013) show that the estimators are (10) and (11).

$$(10)\qquad \hat{\lambda}_0 = \left(\hat{\beta}\hat{\beta}'\right)^{-1}\hat{\beta}\left(\hat{a} + \tfrac{1}{2}\left(\hat{B}^* vec(\hat{\Sigma}) + \sigma^2\iota_N\right)\right), \text{ and}$$

$$(11)\qquad \hat{\lambda}_1 = \left(\hat{\beta}\hat{\beta}'\right)^{-1}\hat{\beta}\hat{c}.$$

[18] As shown in ACM (2013), (3) and (5) are inserted into (2).

On this basis, we can estimate a model-based zero coupon yield curve based on the estimated parameters. One assumption is that log bond prices are linear combinations of the state variables as shown in (12).

(12) $\ln P_t^n = A_n + B_n' X_t + e_t^n$

Given equation (5), excess return can therefore also be formulated as

(13) $rx_{t+1}^{(n-1)} = A_{n-1} + B_{n-1}' X_{t+1} + u_t^{(n-1)} - A_n - B_n' X_t - u_t^n + A_1 + B_1' X_t + u_t^{(1)},$

where

(14) $-r_t = A_1 + B_1' X_t + u_t^{(1)}.$

The expression for excess return, ie (13), is combined with the expression for the excess return generating process in equation (6), which gives us:

(15) $$A_{n-1} + B_{n-1}'(\mu + \phi X_t + v_{t+1}) + u_t^{(n-1)} - A_n - B_n' X_t - u_t^n + A_1 + B_1' X_t + u_t^{(1)}$$

$$= \beta^{(n-1)\prime}(\lambda_0 + \lambda_1 X_t + v_{t+1}) - \frac{1}{2}\left(\beta^{(n-1)\prime} \Sigma \beta^{(n-1)} + \sigma^2\right) + e_{t+1}^{(n-1)}$$

For period n = 1, values for A and B are $A_1 = -\delta_0$ and $B_1' = -\delta'_1$. This generates a recursive system of linear restrictions for the bond pricing parameters.

(16) $A_n = A_{n-1} + B_{n-1}'(\mu - \lambda_0) + \frac{1}{2}(B_{n-1}' \Sigma B_{n-1} + \sigma^2) - \delta_0,$

(17) $B_n' = B_{n-1}'(\Phi - \lambda_1) - \delta'_1,$

(18) $A_0 = 0, B_0' = 0,$

(19) $\beta^{(n)\prime} = B_n'$

(20) $u_t^{(n-1)} - u_t^n + u_t^{(1)} = e_{t+1}^{(n-1)}$

The equations are first estimated as described above and provide the model's price parameters $A_n$ and $B_n'$. On the basis of these estimates, the zero coupon yields $-\frac{1}{n}(A_n + B_n' X_t)$ are estimated. The risk pricing parameters $\lambda_0$ and $\lambda_1$ are set equal to zero and the equations are re-estimated.

This generates risk neutral pricing parameters $A_n^{RN}$ and $B_n^{RN\prime}$. The risk neutral zero coupon yields $-\frac{1}{n}(A_n^{RN} + B_n^{RN\prime} X_t)$ can then be understood as an average of the expected future short-term yields over the period.

The term premium is calculated as the difference between the actual and the risk neutral zero-coupon yields, ie:

(21) $TP_t^n = -\frac{1}{n}(A_n + B_n' X_t) - \left(-\frac{1}{n}\left(A_n^{RN} + B_n^{RN'} X_t\right)\right)$

## Stability of the principal components

An important assumption in the ACM model is the reversion of the term structure of interest rates to a historical average over time. The evolution of the factors is modelled according to 3.1. For the factors to revert over time, this system must be stable.

The stability condition for 3.1 is that all five eigenvalues in the matrix $\phi$ are within the unit circle. Alternatively, all eigenvalues can be said to have a modulus of less than 1. Our estimation results in the following eigenvalues for $\phi$ (Table 1).

**Table 1: Eigenvalues**

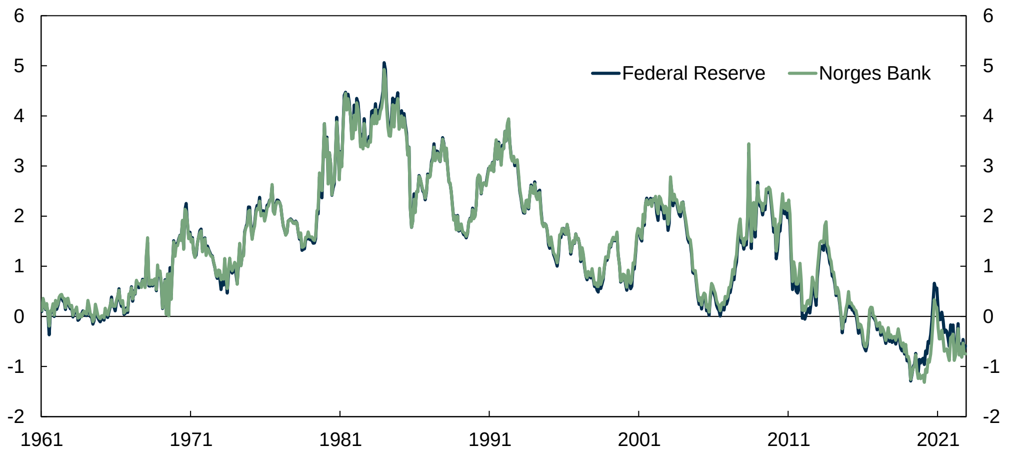
| Eigenvalue | Complex value | Modulus |
|---|---|---|
| $\gamma_1$ | 0.6496+0.0i | 0.6496 |
| $\gamma_2$ | 0.9771+0.0i | 0.9771 |
| $\gamma_3$ | 0.9085+0.0798i | 0.9120 |
| $\gamma_4$ | 0.9085-0.0798i | 0.9120 |
| $\gamma_5$ | 0.92164+0.0i | 0.9216 |

## Reconstruction of the 10-year US term premium

The Federal Reserve publishes both zero coupon yields and term premiums at a daily frequency[19]. We can therefore perform cross-checks to ensure that our results harmonise with the Federal Reserve's calculations.

We convert the Federal Reserve's zero coupon yields to a monthly frequency before estimating term premiums for a 10-years maturity. Chart A shows the estimated 10-year term premium in US yields and our own estimation. Some difference between the curves is expected as Norges Bank's estimation uses a monthly frequency whereas the Federal Reserve uses a daily frequency. The chart shows that differences between the methods are mostly marginal.

**Chart A: Cross-check**
Percentage points. June 1961 – April 2022




Source: Norges Bank

[19] See Federal Reserve (2023a) and Federal Reserve (2023b).

## Deciding on the number of factors

Term premium analysis assumes that the yield curve can be represented by a number of factors estimated using principal component analysis (PCA).

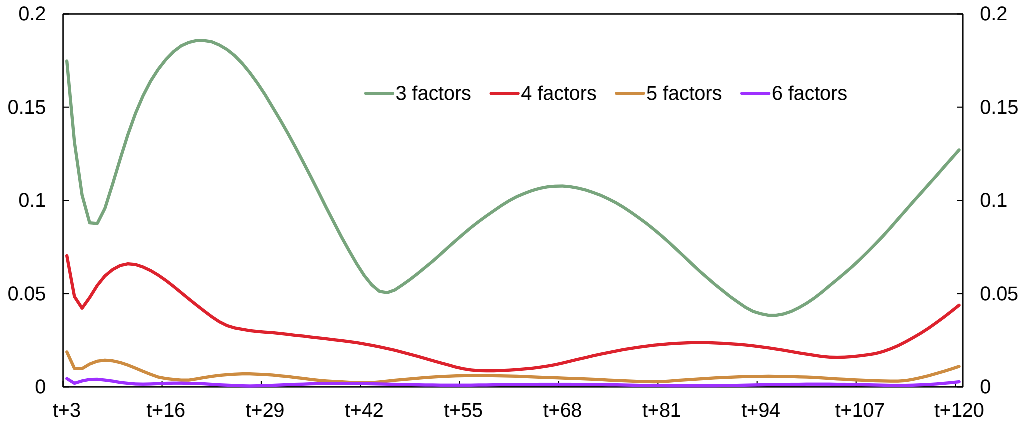
This type of analysis often uses three factors, which can then be interpreted as level, slope and curvature.

Factor interpretation is not as obvious if the number of factors exceeds three but may be appropriate if a higher number strengthens the model's explanatory power. However, additional factors will increase the risk of overfitting the model.

Adrian et al (2008) conclude that the five-factor model is best suited for their data. Bank of England (2014) uses four factors. Jennison (2017) also uses five factors for Australian data. Callahan (2019) applies three factors for New Zealand yields.

The chart below shows the root-mean-square error (RMSE) when using three, four, five and six factors at each of the maturities included in the calculation. It measures the quadratic deviation between actual and estimated yield curves. A higher numerical value indicates a higher deviation of the estimated curve from the actual curve.

**Chart B1: RMSE for models with three, four, five and six factors**



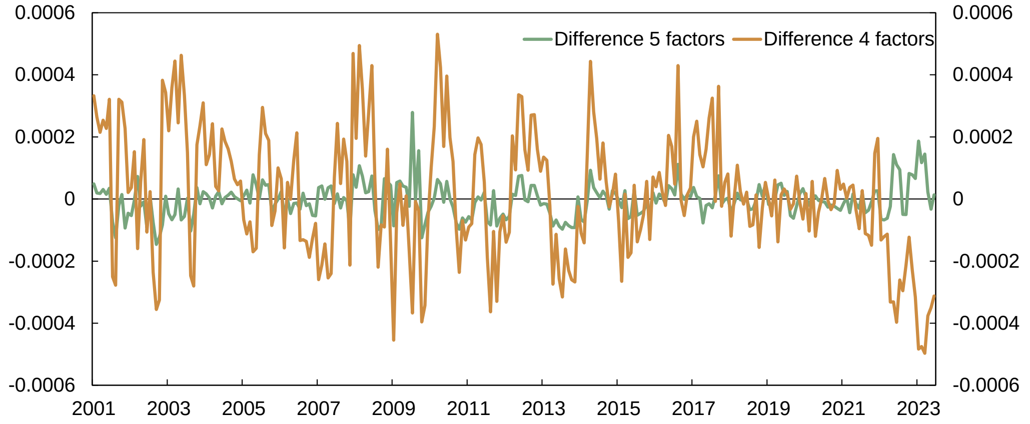

Source: Norges Bank

Chart B shows that the model using three factors estimates actual yields less accurately than the three other models. There is however little difference between the models using three or more factors. Based on an overall assessment, five factors are used in our estimates.

Charts C and D show the difference between actual and estimated yields, generated by a model using four and five factors for three-month and ten-year maturities, respectively. The charts show that the five-factor model produces more accurate projections in two periods, following the financial crisis and the recent period of rising interest rates.

**Chart B2: Difference between actual and estimated yields using four- and five-factor models**

Ten-year maturity




Source: Norges Bank